

# Hedetniemi's Conjecture Via Alternating Chromatic Number

Meysam Alishahi<sup>†</sup> and Hossein Hajiabolhassan<sup>\*</sup>

<sup>†</sup> *Department of Mathematics*

*Shahrood University of Technology, Shahrood, Iran*

`meysam_alishahi@shahroodut.ac.ir`

<sup>\*</sup> *Department of Applied Mathematics and Computer Science*

*Technical University of Denmark*

*DK-2800 Lyngby, Denmark*

<sup>\*</sup> *Department of Mathematical Sciences*

*Shahid Beheshti University, G.C.*

*P.O. Box 19839-63113, Tehran, Iran*

`hhaji@sbu.ac.ir`

## Abstract

In an earlier paper, the present authors (2013) [1] introduced the alternating chromatic number for hypergraphs and used Tucker's Lemma, an equivalent combinatorial version of the Borsuk-Ulam Theorem, to show that the alternating chromatic number is a lower bound for the chromatic number. In this paper, we determine the chromatic number of some families of graphs by specifying their alternating chromatic number. We define matching-dense graph as a graph whose vertex set is the set of all matchings of a specified size of a dense graph and two vertices are adjacent if the corresponding matchings are edge-disjoint. We determine the chromatic number of matching-dense graphs in terms of the generalized Turán number of matchings. Also, we consider Hedetniemi's conjecture which asserts that the chromatic number of the Categorical product of two graphs is equal to the minimum of their chromatic numbers. By topological methods, it has earlier been shown that Hedetniemi's conjecture holds for any two graphs of the family of Kneser graphs, Schrijver graphs, and the iterated Mycielskian of any such graphs. We extend these results to other graphs such as a large family of Kneser multigraphs, matching graphs, and permutation graphs.

**Keywords:** Chromatic Number, Hedetniemi's Conjecture, General Kneser Hypergraph, Turán Number.

**Subject classification:** 05C15

## 1 introduction

For a hypergraph  $H$ , the vertex set of the general Kneser graph  $KG(H)$  is the set of all hyperedges of  $H$  and two vertices are adjacent if the corresponding hyperedges are disjoint. It is known that any graph can be represented by a general Kneser graph. In this paper, we can see that a graph has various representations, nevertheless we can determine the chromatic number of some graphs, by choosing an appropriate representation for them. In view of Tucker's Lemma, an equivalent combinatorial version of the Borsuk-Ulam Theorem, the present authors [1, 2] introduced the alternating chromatic number for hypergraphs which provides a tight

---

<sup>1</sup>The research of Hossein Hajiabolhassan is supported by ERC advanced grant GRACOL.

lower bound for the chromatic number of hypergraphs. Next, they determined the chromatic number of Kneser multigraphs, multiple Kneser graphs and a family of matching graphs, see [1, 2]. A matching graph and a Kneser multigraph can be represented by general Kneser graphs  $KG(H)$  and  $KG(G)$ , respectively, such that the hyperedges of  $H$  are corresponding to all matchings of a specified size of a graph and the hyperedges of  $G$  are corresponding to all subgraphs isomorphic to some fixed prescribed simple graphs of a multigraph such that the multiplicity of each edge is at least 2. Note that a Schrijver graph is a matching graph  $KG(H)$ , where the hyper-edge set of  $H$  is corresponding to all matchings of a specified size of a cycle. Hence, by determining the chromatic number of matching graphs, we generalize Schrijver's Theorem. A challenging and interesting problem in graph coloring is Hedetniemi's conjecture which asserts that the chromatic number of the Categorical product of two graphs is the minimum of that of graphs. There are a few general results about the chromatic number of the Categorical product of graphs whose chromatic number is large enough. In view of topological bounds, it has earlier been shown that Hedetniemi's conjecture holds for any two graphs for which the topological bound on the chromatic number is tight, see [8, 22, 25].

This paper is organized as follows. In Section 1, we set up notations and terminologies. In particular, we will be concerned with the definition of the alternating chromatic number and the strong alternating chromatic number and we mention some results about them. In particular, we show that they provide tight lower bounds not only for the chromatic number of graphs but also for some well-known topological parameters. Also, we introduce alternating Turán number which is a generalization of the Turán number. Next, we introduce a lower bound for the chromatic number in terms of the alternating Turán number. In fact, if one can show that the alternating Turán number is the same as the Turán number for a family of graphs, then we can determine the chromatic number of some family of graphs. In Section 2, we determine the chromatic number of any matching-dense graph  $KG(H)$ , where the hyperedges of  $H$  are corresponding to all matchings of a specified size in a dense graph. As a consequence, we determine the chromatic number of permutation graphs provided that the number of vertices is large enough. In Section 3, we generalize the definitions of alternating and strong alternating chromatic number of graphs and in view of an appropriate representation for the Mycielskian of a graph, we show that the generalized alternating chromatic number behave like the chromatic number for the Mycielskian of a graph. Precisely, we show that the generalized alternating chromatic number of the Mycielskian of a graph  $G$  is at least the generalized alternating chromatic number of  $G$  plus one. Also, we show that the generalized strong alternating chromatic number satisfies Hedetniemi's conjecture. By topological methods, it has earlier been shown that Hedetniemi's conjecture holds for any two graphs of the family of Kneser graphs, Schrijver graphs, and the iterated Mycielskian of any such graphs. We extend this result to other graphs such as a large family of Kneser multigraphs, matching graphs, and permutation graphs.

## 2 Notations and Terminologies

In this section, we setup some notations and terminologies. Hereafter, the symbol  $[n]$  stands for the set  $\{1, 2, \dots, n\}$ . A *hypergraph*  $H$  is an ordered pair  $(V(H), E(H))$ , where  $V(H)$  is a set of elements called *vertices* and  $E(H)$  is a set of nonempty subsets of  $V(H)$  called *hyperedges*. Unless otherwise stated, we consider simple hypergraphs, i.e.,  $E(H)$  is a family of distinct nonempty subsets of  $V(H)$ . A *vertex cover*  $T$  of  $H$  is a subset of its vertices such that each hyperedge of  $H$  is incident to some vertex of  $T$ . Also, a *k-coloring* of a hypergraph  $H$  is a mapping  $h : V(H) \rightarrow [k]$  such that for any hyperedge  $e$ , we have  $|\{h(v) : v \in e\}| \geq 2$ . In other words, no hyperedge is monochromatic. The minimum  $k$  such that there exists a coloring  $h : V(H) \rightarrow [k]$  is called the *chromatic number* of  $H$  and is denoted by  $\chi(H)$ . Note that if the hypergraph  $H$  has some hyperedge with cardinality 1, then there is no  $k$ -coloring for any  $k$ . Therefore, we define the chromatic number of such a hypergraph to be infinite. A hypergraph  $H$  is *k-uniform*, if all hyperedges of  $H$  have the same size  $k$ . By a graph  $G$ , we mean a 2-uniform hypergraph. Also, let  $o(G)$  denote the number of odd components of  $G$ .

For two graphs  $G$  and  $H$ , a mapping  $f : V(G) \rightarrow V(H)$  is called a *homomorphism* from  $G$  to  $H$ , if it preserves the adjacency, i.e., if  $xy \in E(G)$ , then  $f(x)f(y) \in E(H)$ . For brevity, we use  $G \rightarrow H$  to denote that there is a homomorphism from  $G$  to  $H$ . If we have both  $G \rightarrow H$  and  $H \rightarrow G$ , then we say  $G$  and  $H$  are *homomorphically equivalent* and show this by  $G \longleftrightarrow H$ . Note that  $\chi(G)$  is the minimum integer  $k$  for which there is a homomorphism from  $G$  to the complete graph  $K_k$ . An *isomorphism* between  $G$  and  $H$  is a bijection map  $f : V(G) \rightarrow V(H)$  such that both  $f$  and  $f^{-1}$  are homomorphism. For brevity, we use  $G \cong H$  to mention that there is an isomorphism between  $G$  and  $H$ . Also, if  $G \cong H$ , then we say  $G$  and  $H$  are *isomorphic*. For a subgraph  $H$  of  $G$ , the subgraph  $G \setminus H$  is obtained from  $G$  by deleting the edge set of  $H$ . Also,  $G - H$  is obtained from  $G$  by deleting the vertices of  $H$  with their incident edges.

The *general Kneser graph*  $\text{KG}(H)$  has  $E(H)$  as vertex set and two vertices are adjacent if the corresponding hyperedges are disjoint. It is a well-known result that for any graph  $G$ , there is some hypergraph  $H$  such that  $\text{KG}(H) \cong G$ .

A subset  $S \subseteq [n]$  is said to be *s-stable* if  $s \leq |i - j| \leq n - s$  for any distinct  $i, j \in S$ . Hereafter, for a subset  $A \subseteq [n]$ , the symbols  $\binom{A}{k}$  and  $\binom{A}{k}_s$  stand for the set of all  $k$ -subsets and all  $s$ -stable  $k$ -subsets of  $A$ , respectively. The *Kneser graph*  $\text{KG}(n, k)$  and *s-stable Kneser graph*  $\text{KG}(n, k)_{s\text{-stab}}$  have all  $k$ -subsets and all  $s$ -stable  $k$ -subsets of  $[n]$  as their vertex sets, respectively. Also, in these two graphs, two vertices are adjacent, when the corresponding sets are disjoint. If we set  $H_1 = \left([n], \binom{[n]}{k}\right)$  and  $H_2 = \left([n], \binom{[n]}{k}_s\right)$ , then  $\text{KG}(H_1) \cong \text{KG}(n, k)$  and  $\text{KG}(H_2) \cong \text{KG}(n, k)_{s\text{-stab}}$ . The graph  $\text{KG}(n, k)_{2\text{-stab}} = \text{SG}(n, k)$  is a well-known graph called *Schrijver graph*. In 1955, Kneser conjectured that  $\chi(\text{KG}(n, k)) = n - 2k + 2$ . Lovász [17], by using the Borsuk-Ulam theorem, proved this conjecture. Next, this was improved by Schrijver [20] who proved that the Schrijver graph  $\text{SG}(n, k)$  is a critical subgraph of the Kneser graph  $\text{KG}(n, k)$  with the same chromatic number.

## 2.1 Alternating Chromatic Number

Suppose that  $H \subseteq 2^{[n]}$  is a hypergraph, where  $n$  is a positive integer. Consider  $X \in \{-1, 0, +1\}^n \setminus \{(0, 0, \dots, 0)\}$ . Define  $X^+ = \{i \in [n] : x_i = +1\}$  and  $X^- = \{i \in [n] : x_i = -1\}$ . By abuse of notation, we set  $X = (X^+, X^-)$ . Throughout this paper, We use interchangeably these two kinds of presentations of  $X$ , i.e.,  $X = (x_1, \dots, x_n)$  or  $X = (X^+, X^-)$ . The sequence  $x_1, x_2, \dots, x_m \in \{-1, +1\}$  is said to be an *alternating sequence*, if any two consecutive terms are different. For any  $X = (x_1, x_2, \dots, x_n) \in \{-1, 0, +1\}^n \setminus \{(0, 0, \dots, 0)\}$ , the *alternating number* of  $X$ ,  $alt(X)$ , is the length of a longest alternating subsequence of nonzero terms of  $(x_1, x_1, \dots, x_n)$ . Note that we consider just nonzero entries to determine the alternating number of  $X$ . For a *linear ordering* (or a permutation)  $\sigma = (i_1, i_2, \dots, i_n)$  of  $[n]$ , define  $\sigma(j) = i_j$ , where  $1 \leq j \leq n$ . Also, we sometimes use the usual notation for a linear ordering of  $[n]$ , i.e.,  $\sigma : i_1 < i_2 < \dots < i_n$ , and we use interchangeably these two kinds of presentations of any linear ordering. For a *linear ordering*  $\sigma = (i_1, i_2, \dots, i_n)$  of  $[n]$ , set  $X_\sigma^+ = \{\sigma(i) \in [n] : x_i = +1\}$ ,  $X_\sigma^- = \{\sigma(i) \in [n] : x_i = -1\}$ , and  $X_\sigma = (X_\sigma^+, X_\sigma^-)$ .

For any hypergraph  $H \subseteq 2^{[n]}$  and  $\sigma \in S_n$ , define  $alt_\sigma(H)$  (resp.  $salt_\sigma(H)$ ) to be the largest integer  $k$  such that there exists an  $X \in \{-1, 0, +1\}^n \setminus \{(0, 0, \dots, 0)\}$  with  $alt(X) = k$  and that none (resp. at least one) of  $X_\sigma^+$  and  $X_\sigma^-$  contains any hyperedge of  $H$ . If for each  $X \in \{-1, 0, +1\}^n \setminus \{(0, 0, \dots, 0)\}$ , either  $X_\sigma^+$  or  $X_\sigma^-$  has some hyperedge of  $\mathcal{F}$ , then we define  $alt_\sigma(\mathcal{F}) = 0$ . Note that  $alt_\sigma(H) \leq salt_\sigma(H)$  and equality can hold. Now, define  $alt(H) = \min\{alt_\sigma(H) : \sigma \in S_n\}$  and  $salt(H) = \min\{salt_\sigma(H) : \sigma \in S_n\}$ . Now, we define the *alternating chromatic number* and *strong alternating chromatic number* of a graph  $G$ , respectively, as follows

$$\chi_{alt}(G) = \max_{\mathcal{F}} \{|V(\mathcal{F})| - alt(\mathcal{F}) : KG(\mathcal{F}) \cong G\}$$

and

$$\chi_{salt}(G) = \max_{\mathcal{F}} \{|V(\mathcal{F})| + 1 - salt(\mathcal{F}) : KG(\mathcal{F}) \cong G\}.$$

**Remark 1.** For any hypergraph  $H$  on  $n$  vertices, in view of the definition of  $alt_\sigma(H)$  where  $\sigma$  is an ordering of the vertex set of  $H$ , throughout this paper, we assume that  $V(H)$  was identified with the set  $[n]$ . We may represent  $V(H)$  with different sets, nevertheless we can consider any representation as a relabeling of the set  $[n]$ .

It was proved in [1, 2] that both alternating chromatic number and strong alternating chromatic number provide tight lower bounds for the chromatic number of graphs.

**Theorem A.** [1] For any graph  $G$ , we have

$$\chi(G) \geq \max\{\chi_{alt}(G), \chi_{salt}(G)\}.$$

In the sequel, we introduce another proof for the aforementioned theorem. In fact, we show that the alternating chromatic number and the strong alternating

chromatic number provide tight lower bounds for some well-known topological parameters. We refer the reader to [18, 21] for some basic definitions of algebraic topology such as two kinds of box complexes  $B(G)$  and  $B_0(G)$ . There are several charming topological lower bounds for the chromatic number of an arbitrary graph  $G$  as follows (see [18, 21])

$$\chi(G) \geq \text{ind}(B(G)) + 2 \geq \text{ind}(B_0(G)) + 1 \geq \text{coind}(B_0(G)) + 1 \geq \text{coind}(B(G)) + 2.$$

The proof of the next lemma is based on ideas similar to those used in an interesting proof of Ziegler (see page 67 in [18]) for Gale's Lemma and Proposition 8 of [21].

**Lemma 1.** *For any graph  $G$ , the following inequalities hold.*

- a)  $\chi(G) \geq \text{coind}(B_0(G)) + 1 \geq \chi_{alt}(G),$
- b)  $\chi(G) \geq \text{coind}(B(G)) + 2 \geq \chi_{salt}(G).$

**Proof.** Let  $\mathcal{F} \subseteq 2^{[n]}$  and that  $\text{KG}(\mathcal{F})$  is isomorphic to  $G$  such that there exists a  $\sigma \in S_n$  with  $\chi_{alt}(G) = n - \text{alt}_\sigma(\text{KG}(\mathcal{F}))$ . Note that  $\mathcal{F}$  may have some isolated vertices and they seem like extra notations. Although, note that isolated vertices are sometimes useful to introduce an appropriate ordering to determine the alternating chromatic number. Set  $m = \chi_{alt}(G) - 1$ . Consider the following curve

$$\gamma = \{(1, t, t^2, \dots, t^m) \in \mathbb{R}^{m+1} : t \in \mathbb{R}\}.$$

Set  $W = \{w_1, w_2, \dots, w_n\}$  where  $w_i = \gamma(i)$ , for  $i = 1, 2, \dots, n$ . Now, consider the subset  $\{v_1, v_2, \dots, v_n\}$  of points of  $S^m$ , where  $v_i = (-1)^i \frac{W_i}{\|W_i\|}$ , for any  $1 \leq i \leq n$ . Now, suppose that  $\sigma(i) \in [n]$  is identified with  $v_i$ , for any  $1 \leq i \leq n$ . It can be checked that every hyperplane passing through the origin intersects  $\gamma$  in no more than  $m$  points. Moreover, if a hyperplane intersects the curve in exactly  $m$  points, then the hyperplane cannot be tangent to the curve; and consequently, at each intersection point, the curve passes from one side of the hyperplane to the other side.

Now, we show that for any  $x \in S^m$ , the open hemispheres  $H(x)$  or  $H(-x)$  contains some member of  $\mathcal{F}$ . On the contrary, suppose that there is an  $x \in S^m$  such that neither  $H(x)$  nor  $H(-x)$  contains any member of  $\mathcal{F}$ . Let  $h$  be the hyperplane passing through the origin which contains the boundary of  $H(x)$ . We can move this hyperplane continuously to a position such that it still contains the origin and has exactly  $m$  points of  $W = \{w_1, w_2, \dots, w_n\}$  while during this movement no points of  $W$  crosses from one side of  $h$  to the other side. Consequently, during the aforementioned movement, no points of  $V = \{v_1, v_2, \dots, v_n\}$  crosses from one side of  $h$  to the other side. Therefore, without loss of generality, we may assume that  $h$  intersects  $W$  at exactly  $m$  points of  $W$ . Hence, at each of this intersections,  $\gamma$  passes from one side of  $h$  to the other side. Assume that  $h^+$  and  $h^-$  be two open half-spaces determined by the hyperplane  $h$ . Now, consider  $X = (x_1, x_2, \dots, x_n) \in \{-1, 0, +1\}^n \setminus \{(0, 0, \dots, 0)\}$  such that

$$x_i = \begin{cases} 0 & \text{if } w_i \text{ is on } h \\ 1 & \text{if } w_i \text{ is in } h^+ \text{ and } i \text{ is odd} \\ 1 & \text{if } w_i \text{ is in } h^- \text{ and } i \text{ is even} \\ -1 & \text{otherwise.} \end{cases}$$

Assume that  $x_{i_1}, x_{i_2}, \dots, x_{i_{n-m}}$  are nonzero entries of  $X$ , where  $i_1 < i_2 < \dots < i_{n-m}$ . It is easy to check that any two consecutive terms of  $x_{i_j}$ 's have different signs. Since  $X$  has  $n - m = \text{alt}_\sigma(\mathcal{F}) + 1$  nonzero entries, we have  $\text{alt}(X) = \text{alt}_\sigma(\mathcal{F}) + 1$ ; and therefore, either  $X_\sigma^+$  or  $X_\sigma^-$  has some member of  $\mathcal{F}$ . Now, one see that it implies that either  $H(x)$  or  $H(-x)$  has some member of  $\mathcal{F}$ . For any vertex  $A$  of  $\text{KG}(\mathcal{F})$  and any  $x \in S^m$ , define  $D_A(x)$  to be the smallest distance of a point in  $A \subset S^m$  from the set  $S^m \setminus H(x)$ . Note that  $D_A(x) > 0$  if and only if  $H(x)$  contains  $A$ . Define

$$D(x) = \sum_{A \in \mathcal{F}} (D_A(x) + D_A(-x)).$$

For any  $x \in S^m$ , either  $H(x)$  or  $H(-x)$  has some member of  $\mathcal{F}$ ; and therefore,  $D(x) > 0$ . Thus, the map

$$f(x) = \frac{1}{D(x)} \left( \sum_{A \in \mathcal{F}} D_A(x) \|(A, 1)\| + \sum_{A \in \mathcal{F}} D_A(-x) \|(A, 2)\| \right)$$

from  $S^m$  to  $\|B_0(\text{KG}(\mathcal{F}))\|$  is a  $\mathbb{Z}_2$ -map. It implies  $\text{coind}(B_0(G)) \geq m$ .

To prove the second part, assume that  $\mathcal{G} \subseteq 2^{[n]}$  and that  $\text{KG}(\mathcal{G})$  is isomorphic to  $G$  such that there exists a  $\sigma \in S_n$  with  $\chi_{\text{slat}}(G) = n + 1 - \text{salt}_\sigma(\mathcal{G})$ . Now, in the proof of part (a), we set  $m = \chi_{\text{slat}}(G) - 2$ . With the same argument as in the proof of part (a), one can see that for any  $x \in S^m$ , both  $H(x)$  and  $H(-x)$  have some member of  $\mathcal{G}$ . Define  $D(x) = \sum_{A \in \mathcal{G}} D_A(x)$ . For any  $x \in S^m$ ,  $H(x)$  has some member of  $\mathcal{G}$ ; and therefore,  $D(x) > 0$ . Thus, the map

$$f(x) = \frac{1}{2D(x)} \sum_{A \in \mathcal{G}} D_A(x) \|(A, 1)\| + \frac{1}{2D(-x)} \sum_{A \in \mathcal{G}} D_A(-x) \|(A, 2)\|$$

from  $S^m$  to  $\|B(\text{KG}(\mathcal{G}))\|$  is a  $\mathbb{Z}_2$ -map. It implies  $\text{coind}(B(G)) \geq m$ . ■

In [1, 2], the chromatic number of several families of graphs was determined, by computing their alternating chromatic number or their strong alternating chromatic number. Note that, by considering the natural ordering of positive integers, one can see that  $\chi(\text{SG}(n, k)) = \chi_{\text{salt}}(\text{SG}(n, k)) = n - 2k + 2$ . Now, we show that  $\chi(\text{SG}(n, 2)) = \chi_{\text{alt}}(\text{SG}(n, 2)) = n - 2$  and  $\chi(\text{SG}(2k+1, k)) = \chi_{\text{alt}}(\text{SG}(2k+1, k)) = 3$ . Note that  $\chi_{\text{alt}}(\text{SG}(n, 2)) \leq \chi(\text{SG}(n, 2))$  and also it is simple to show  $\chi(\text{SG}(n, 2)) \leq n - 2$ . Hence, it is sufficient to show  $n - 2 \leq \chi_{\text{alt}}(\text{SG}(n, 2))$ . To see this, first assume that  $n = 2t$ . Consider a 2-uniform hypergraph  $H = (V(H), E(H))$ , where  $V(H) = \{1, 2, \dots, 4t\} = \{1, 2, \dots, 2t, a_1, \dots, a_{2t}\}$  and  $E(H) = \binom{[2t]}{2}_2$ . Note that  $a_1, \dots, a_{2t}$  are isolated vertices of  $H$ . Consider the ordering  $\sigma$  for  $V(H)$  as follows

$$\begin{aligned} \sigma : 1 &< a_1 < t+1 < a_2 < 2 < a_3 < t+2 < a_4 < 3 < a_5 < t+3 < a_6 < 4 < \dots \\ &< a_{2t-4} < t-1 < a_{2t-3} < 2t-1 < a_{2t-2} < t < a_{2t-1} < 2t < a_{2t} \end{aligned}$$

For any vector  $X$ , set  $l(X)$  to be the number of nonzero terms of  $X$ . Now, we show that if  $X = (x_1, \dots, x_{4t}) \in \{-1, 0, +1\}^{4t} \setminus \{(0, 0, \dots, 0)\}$  and  $\text{alt}(X) = 2t + 3$ , then either  $X_\sigma^+$  or  $X_\sigma^-$  contains some hyperedge of  $H$ . To see this, it is sufficient to



prove it for any  $X$  such that  $\text{alt}(X) = l(X) = 2t + 3$ . Moreover, one can suppose that for any even integer  $i$ , we have  $x_i \neq 0$ . Since otherwise, one can check that there exists a  $Y = (y_1, \dots, y_{4t}) = (Y^+, Y^-) \in \{-1, 0, +1\}^{4t} \setminus \{(0, 0, \dots, 0)\}$  such that  $\text{alt}(Y) = l(Y) = 2t + 3$ ,  $Y_\sigma^+ \cap \{1, 2, \dots, 2t\} \subseteq X_\sigma^+ \cap \{1, 2, \dots, 2t\}$ ,  $Y_\sigma^- \cap \{1, 2, \dots, 2t\} \subseteq X_\sigma^- \cap \{1, 2, \dots, 2t\}$ , and  $y_j \neq 0$  for any even integer  $j$ . Consequently, if we show that either  $Y_\sigma^+$  or  $Y_\sigma^-$  contains some hyperedge of  $H$ , then the same assertion holds for  $X_\sigma^+$  or  $X_\sigma^-$ . Now, since  $\text{alt}(X) = l(X) = 2t + 3$  and for any even integer  $i$ , we have  $x_i \neq 0$ , one can see that there are exactly 3 odd integers for which their coordinates in  $X$  are nonzero. Also, one can see that for any two odd integers  $i$  and  $j$ , if  $|\sigma(i) - \sigma(j)| = 1$  or  $|\sigma(i) - \sigma(j)| = 2t - 1$ , then either  $|i - j| = 4$  or  $i \in \{1, 3\}$ . Now, we claim that there is no odd integer  $1 \leq i \leq 4t - 5$ , such that  $x_i = x_{i+4} = +1$  (resp.  $x_i = x_{i+4} = -1$ ). Otherwise, since  $\text{alt}(X) = l(X) = 2t + 3$  and that for any even integer  $j$ , we have  $x_j \neq 0$ , one can see that  $x_{i+2} = +1$  (resp.  $x_{i+2} = -1$ ). In this case, one can see that  $X_\sigma^+$  (resp.  $X_\sigma^-$ ) contains some hyperedge of  $H$ . Also, for a contradiction, suppose  $x_1 = x_{4t-1} = +1$  (resp.  $x_1 = x_{4t-1} = -1$ ). If we consider the other nonzero coordinate  $x_j \neq 0$  ( $j$  is an odd integer), then, in view of  $\text{alt}(X) = 2t + 3$ ,  $x_j$  should have the same sign with exactly one of  $x_1$  or  $x_{4t-1}$  which is impossible. Similarly, if  $x_3 = x_{4t-3} = +1$  (resp.  $x_3 = x_{4t-3} = -1$ ), then one can prove the assertion.

Also, for  $n = 2t + 1$ , consider a 2-uniform hypergraph  $H = (V(H), E(H))$ , where  $V(H) = \{1, 2, \dots, 4t + 2\} = \{1, 2, \dots, 2t + 1, a_1, \dots, a_{2t+1}\}$  and  $E(H) = \binom{[2t+1]}{2}_2$ . Also, consider the ordering  $\sigma$  as follows

$$\begin{aligned} \sigma : t + 1 &< a_1 < 1 < a_2 < t + 2 < a_3 < 2 < a_4 < t + 3 < a_5 < 3 < a_6 < t + 4 < \dots \\ &< a_{2t-3} < t - 1 < a_{2t-2} < 2t < a_{2t-1} < t < a_{2t} < 2t + 1 < a_{2t+1} \end{aligned}$$

Similarly, one can show  $\chi(\text{SG}(n, 2)) = \chi_{\text{alt}}(\text{SG}(n, 2)) = n - 2$ .

In [2], it was shown that  $\chi(\text{SG}(5, 2)) = \chi_{\text{alt}}(\text{SG}(5, 2)) = 3$ . Similarly, one can see that  $\chi(\text{SG}(2k + 1, k)) = \chi_{\text{alt}}(\text{SG}(2k + 1, k)) = 3$ . To see this, consider a  $k$ -uniform hypergraph  $H = (V(H), E(H))$ , where  $V(H) = \{1, 2, \dots, 4k + 2\} = \{1, 2, \dots, 2k + 1, a_1, \dots, a_{2k+1}\}$  and  $E(H) = \binom{[2k+1]}{k}_2$ . Note that  $a_1, \dots, a_{2k+1}$  are isolated vertices of  $H$ . Consider the ordering  $\sigma$  for  $V(H)$  as follows

$$1 < a_1 < 3 < a_2 < \dots < 2k + 1 < a_{k+1} < 2 < a_{k+2} < 4 < a_{k+3} < \dots < 2k < a_{2k+1}.$$

One can check that  $\chi(\text{SG}(2k + 1, k)) = \chi_{\text{alt}}(\text{SG}(2k + 1, k)) = 3$ .

## 2.2 Alternating Turán Number

A *hypergraph labeling* is an assignment of labels, from a set of symbols, to the hyperedges or vertices, or both, of a hypergraph. Formally, given a hypergraph  $H$ , a vertex (resp. hyperedge) labeling is a function from the vertices (resp. hyperedges) of  $H$  to a set of labels. For a family  $\mathcal{F}$  of unlabeled hypergraphs and an unlabeled hypergraph  $H$ , a subhypergraph of  $H$  is an  $\mathcal{F}$ -*subhypergraph*, if it is a member of  $\mathcal{F}$ . Moreover, for a labeled hypergraph  $H$  and a family  $\mathcal{F}$  of labeled hypergraphs, a subhypergraph of  $H$  is an  $\mathcal{F}$ -*subhypergraph*, if there is an isomorphism  $f$  between this subhypergraph and a member of  $\mathcal{F}$  such that  $f$  preserves labels. Note that any directed graph can be considered as a labeled graph. *The general Kneser hypergraph*

$\text{KG}(H, \mathcal{F})$  (resp.  $\text{KG}_v(H, \mathcal{F})$ ) has all  $\mathcal{F}$ -subhypergraphs of  $H$  as vertex set and two vertices are adjacent if the corresponding  $\mathcal{F}$ -subhypergraphs are hyperedge-disjoint (resp. vertex-disjoint). Hereafter, unless otherwise stated, we consider unlabeled hypergraph. A hypergraph  $H$  is said to be  $\mathcal{F}$ -free, if it has no subhypergraph isomorphic to a member of  $\mathcal{F}$ . For a hypergraph  $G$ , define  $\text{ex}(G, \mathcal{F})$ , the *generalized Turán number of the family  $\mathcal{F}$  in the hypergraph  $G$* , to be the maximum number of hyperedges of an  $\mathcal{F}$ -free spanning subhypergraph  $H$  of  $G$ . A subhypergraph of  $G$  is called  $\mathcal{F}$ -extremal if it has  $\text{ex}(G, \mathcal{F})$ -hyperedges and it is  $\mathcal{F}$ -free. We denote the family of all  $\mathcal{F}$ -extremal subhypergraphs of  $G$  with  $EX(G, \mathcal{F})$ . It is usually a hard problem to determine the exact value of  $\text{ex}(G, \mathcal{F})$ . The concept of Turán number was generalized in [2] in order to find the chromatic number of some families of graphs as follows. Let  $H$  be a hypergraph with  $E(H) = \{e_1, e_2, \dots, e_m\}$ . For any ordering  $\sigma = (e_{i_1}, e_{i_2}, \dots, e_{i_m})$ , a 2-coloring of a subset of hyperedges of  $H$  (with 2 colors red and blue) is said to be *alternating* (respect to the ordering  $\sigma$ ), if any two consecutive colored edges (in the ordering  $\sigma$ ) receive different colors. Note that we may assign no color to some hyperedges of  $H$ . In other words, in view of the ordering  $\sigma$ , we assign two colors red and blue alternatively to a subset of hyperedges of  $H$ . The maximum number of hyperedges of  $H$  that can be colored alternatively (respect to the ordering  $\sigma$ ) by 2-colors such that each (resp. at least one of) color class has no member of  $\mathcal{F}$ , is denoted by  $\text{ex}_{alt}(H, \mathcal{F}, \sigma)$  (resp.  $\text{ex}_{salt}(H, \mathcal{F}, \sigma)$ ). Now, we are in a position to define the *alternating Turán number* and *strong alternating Turán number* of the family  $\mathcal{F}$  in the hypergraph  $H$ ,  $\text{ex}_{alt}(H, \mathcal{F})$  and  $\text{ex}_{salt}(H, \mathcal{F})$ , respectively, as follows

$$\text{ex}_{alt}(H, \mathcal{F}) = \min \{ \text{ex}_{alt}(H, \mathcal{F}, \sigma); \sigma \in S_{E(H)} \}$$

and

$$\text{ex}_{salt}(H, \mathcal{F}) = \min \{ \text{ex}_{salt}(H, \mathcal{F}, \sigma); \sigma \in S_{E(H)} \}.$$

For a hypergraph  $H$ , let  $F$  be a member of  $EX(G, \mathcal{F})$  and  $\sigma$  be an arbitrary ordering of  $E(H)$ . Now, if we color the edges of  $F$  alternatively with two colors with respect to the ordering  $\sigma$ , one can see that any color class has no member of  $\mathcal{F}$ ; and therefore,  $\text{ex}(H, \mathcal{F}) \leq \text{ex}_{alt}(H, \mathcal{F}, \sigma)$ . Also, it is clear that if we assign colors to more than  $2\text{ex}(H, \mathcal{F})$  edges, then a color class has at least more than  $\text{ex}(H, \mathcal{F})$  edges. It implies  $\text{ex}_{alt}(H, \mathcal{F}, \sigma) \leq 2\text{ex}(H, \mathcal{F})$ . Consequently,

$$\text{ex}(H, \mathcal{F}) \leq \text{ex}_{alt}(H, \mathcal{F}) \leq 2\text{ex}(H, \mathcal{F}).$$

Next lemma was proved in [2].

**Lemma A.** [2] *For any hypergraph  $H = (V(H), E(H))$  and a family  $\mathcal{F}$  of hypergraphs,*

$$|E(H)| - \text{ex}_{alt}(H, \mathcal{F}) \leq \chi(\text{KG}(H, \mathcal{F})) \leq |E(H)| - \text{ex}(H, \mathcal{F}),$$

$$|E(H)| + 1 - \text{ex}_{salt}(H, \mathcal{F}) \leq \chi(\text{KG}(H, \mathcal{F})) \leq |E(H)| - \text{ex}(H, \mathcal{F}).$$

The previous lemma enables us to find the chromatic number of some families of graphs. Assume that  $\mathcal{F}$  is a family of graphs and  $G$  is a given graph. If we find an



appropriate ordering  $\sigma$  of the edges of  $G$  and prove that we have either  $\text{ex}_{alt}(G, \mathcal{F}) = \text{ex}(G, \mathcal{F})$  or  $\text{ex}_{salt}(G, \mathcal{F}) - 1 = \text{ex}(G, \mathcal{F})$ , then we can conclude that

$$\chi(\text{KG}(G, \mathcal{F})) = |E(G)| - \text{ex}(G, \mathcal{F}).$$

In this regard, in [1, 2], the chromatic number of several families of graphs was computed by this observation.

For a given 2-coloring of a subset of hyperedges of  $H$ , a spanning subhypergraph of  $H$  whose hyperedge set contains all edges such that we have assigned color red (resp. blue) to them, is termed the *red subhypergraph*  $H^R$  (resp. *blue subhypergraph*  $H^B$ ). Furthermore, by abuse of notation, any edge of  $H^R$  (resp.  $H^B$ ) is termed a red edge (resp. blue edge).

### 3 Matching Graphs

In this section, we investigate the chromatic number of graphs via their alternating chromatic number. In [2], for a sparse graph  $G$ , the chromatic number of the matching-sparse graph  $\text{KG}(G, rK_2)$  was studied. In contrast, in the sequel, we determine the chromatic number of the matching graph  $\text{KG}(G, rK_2)$  provided that  $G$  is a dense graph.

#### 3.1 Matching-Dense Graphs

Let  $G$  be a graph with  $V(G) = \{u_1, \dots, u_n\}$ . The graph  $G$  is termed  $(r, c)$ -locally Eulerian, if there are edge-disjoint nontrivial Eulerian subgraphs  $H_1, \dots, H_n$  of  $G$  such that for any  $1 \leq i \leq n$ , we have  $u_i \in H_i$  and that for any  $u \in V(H_i)$ , where  $u \neq u_i$ , we have  $\deg_{H_i}(u) \geq (r-1)\deg_{H_i}(u) + c$ .

**Lemma 2.** *Let  $r \geq 2$  and  $s$  be nonnegative integers. Also, let  $G$  be a graph with  $n$  vertices and  $\delta(G) > \binom{r+2}{2} + (r-2)s$ . If there exists an  $(r+s, c)$ -locally Eulerian graph  $H$  such that  $G$  is a subgraph of  $H$ ,  $|V(H)| = |V(G)| + s$ , and  $c \geq \binom{r-1}{2} + (s+3)(r-1)$ , then  $\chi(\text{KG}(G, rK_2)) = \chi_{alt}(\text{KG}(G, rK_2)) = |E(G)| - \text{ex}(G, rK_2)$ .*

**Proof.** In view of Lemma A, it is sufficient to show that there exists an ordering  $\sigma$  of the edges of  $G$  such that  $\text{ex}_{alt}(G, rK_2, \sigma) = \text{ex}(G, rK_2)$ .

Assume that  $V(G) = \{u_1, \dots, u_n\}$  and  $V(H) = \{u_1, \dots, u_{n+s}\}$ . In view of definition of  $(r, c)$ -locally Eulerian graph, there are pairwise edge-disjoint nontrivial Eulerian subgraphs  $H_1, \dots, H_{n+s}$  of  $H$  such that for any  $1 \leq i \leq n+s$ , we have  $u_i \in H_i$  and that for any  $u \in V(H_i)$ , where  $u \neq u_i$ , we have  $\deg_{H_i}(u) \geq (r+s-1)\deg_{H_i}(u) + \binom{r-1}{2} + (s+3)(r-1)$ .

To find the ordering  $\sigma$ , add a new vertex  $x$  and join it to all vertices of  $H$  by edges with multiplicity two to obtain the graph  $H'$ . Precisely, for any  $1 \leq i \leq n+s$ , join  $x$  and  $u_i$  with two distinct edges  $f_i$  and  $f'_i$ . Now, if  $H'$  has no odd vertices, then set  $\bar{H} = H'$ ; otherwise, add a new vertex  $z$  to  $H'$  and join it to all odd vertices of  $H'$  to obtain the graph  $\bar{H}$ . The graph  $\bar{H}$  is an even graph; and therefore, it has an Eulerian tour.

Also, note that the graph  $K = \bar{H} - x$  is an even graph; and accordingly, any connected component of  $K' = K \setminus \left( \bigcup_{i=1}^{n+s} H_i \right)$  is also an Eulerian subgraph. Without loss of generality, assume that  $K_1, \dots, K_l$  are the connected components of  $K'$ . Construct an Eulerian tour for  $\bar{H}$  as follows. At  $i^{\text{th}}$  step, where  $1 \leq i \leq n+s$ , start from the vertex  $x$  and traverse the edge  $f_i$ . Consider an arbitrary Eulerian tour of  $H_i$  starting at  $u_i$  and traverse it. Next, if there exists a  $1 \leq j \leq l$  such that  $u_i \in K_j$  and the edge set of  $K_j$  is still untraversed, then consider an Eulerian tour for  $K_j$  starting at  $u_i$  and traverse it. Next, traverse the edge  $f'_i$  and if  $i < n+s$ , then start  $(i+1)^{\text{th}}$  step. Construct an ordering  $\sigma$  for the edge set of the graph  $G$  such that the ordering of edges in  $E(G)$  are corresponding to their ordering in the aforementioned Eulerian tour, i.e., if we traverse the edge  $e_i \in E(G)$  before the edge  $e_j \in E(G)$  in the Eulerian tour, then in the ordering  $\sigma$  we have  $e_i < e_j$ .

Now, we claim that  $\text{ex}_{alt}(G, rK_2, \sigma) = \text{ex}(G, rK_2)$ . Note that for any  $r-1$  vertices  $\{u_{i_1}, \dots, u_{i_{r-1}}\} \subseteq V(G)$ , we have

$$\text{ex}(G, rK_2) \geq \sum_{j=1}^{r-1} \deg_G(u_{i_j}) - \binom{r-1}{2}.$$

Consider an alternating 2-coloring of the edges of  $G$  with respect to the ordering  $\sigma$  of length  $\text{ex}(G, rK_2) + 1$ , i.e., we assigned  $\text{ex}(G, rK_2) + 1$  blue and red colors alternatively to the edges of  $G$  with respect to the ordering  $\sigma$ . For a contradiction, suppose that both red spanning subgraph  $G^R$  and blue spanning subgraph  $G^B$  are  $rK_2$ -free subgraphs. In view of the Berge-Tutte Formula, there are two sets  $T^R$  and  $T^B$  such that  $|V(G^R)| - o(G^R - T^R) + |T^R| \leq 2r - 2$  and  $|V(G^B)| - o(G^B - T^B) + |T^B| \leq 2r - 2$ . In view of these inequalities, one can see that  $|T^R| \leq r - 1$  and  $|T^B| \leq r - 1$ . Moreover, the number of edges of  $G^R$  not incident to some vertex of  $T^R$  ( $|E(G^R - T^R)|$ ) is at most  $\binom{2r-2|T^R|-1}{2}$ . To see this, assume that  $O_1^R, O_2^R, \dots, O_{t_R}^R$  are all connected components of  $G^R - T^R$ , where  $t_R \geq o(G^R - T^R) \geq |V(G^R)| + |T^R| - 2r + 2$ . We have

$$\begin{aligned} |E(G^R - T^R)| &\leq \sum_{i=1}^{t_R} \binom{|V(O_i^R)|}{2} \leq \binom{\sum_{i=1}^{t_R} |V(O_i^R)| - (t_R - 1)}{2} \\ &\leq \binom{2r-2|T^R|-1}{2}. \end{aligned}$$

Similarly, we show  $|E(G^B - T^B)| \leq \binom{2r-2|T^B|-1}{2}$ . Also, in view of the ordering  $\sigma$ , any vertex  $u$  of  $G$  is incident to at most  $\frac{1}{2}(\deg_G(u) + s + 3)$  edges of  $G^R$  (resp.  $G^B$ ). First, we show that if either  $|T^R| \leq r - 2$  or  $|T^B| \leq r - 2$ , then the assertion holds.

If  $|T^R| \leq r - 2$ , then

$$\begin{aligned}
|E(G^R)| &\leq \binom{2r - 2|T^R| - 1}{2} + \sum_{u \in T^R} \frac{1}{2}(\deg_G(u) + s + 3) \\
&\leq \binom{2r - 2|T^R| - 1}{2} + \frac{\text{ex}(G, rK_2)}{2} - \frac{r - 1 - |T^R|}{2}\delta(G) + \frac{1}{2}\binom{r-1}{2} + \frac{(s+3)|T^R|}{2} \\
&< \frac{\text{ex}(G, rK_2)}{2}.
\end{aligned}$$

Since  $|E(G^B)| \leq |E(G^R)| + 1$ , we have  $\text{ex}_{alt}(G, rK_2, \sigma) = |E(G^R)| + |E(G^B)| \leq \text{ex}(G, rK_2)$  which is impossible. Similarly, if  $|T^B| \leq r - 2$ , then the assertion holds.

If  $|T^R| = |T^B| = r - 1$ , in view of the Berge-Tutte Formula, one can conclude that all connected components of  $G^R - T^R$  and  $G^B - T^B$  are isolated vertices. This means that  $T^R$  and  $T^B$  are vertex covers of  $G^R$  and  $G^B$ , respectively. Now, we show that  $T^R = T^B$ . One the contrary, suppose that  $T^R \neq T^B$ . Without loss of generality, let  $|E(G^R)| \geq |E(G^B)|$  and choose a vertex  $u_i \in T^R \setminus T^B$ .

Set  $L^B = |E(H_i^B)|$ , i.e., the number of blue edges of  $H_i$ . Since  $T^B$  is a vertex cover of  $G^B$  and all edges of  $H_i$  appear consecutively in the ordering  $\sigma$  (according to an Eulerian tour of  $H_i$ ), one can check that at least  $2L^B$  edges of  $H_i$  are incident to the vertices of  $T^B$ . It implies that there is a vertex  $z \in V(H_i) \setminus \{u_i\}$  with degree at least  $\frac{2L^B}{r-1}$ . However, by the assumption that for any  $u \in V(H_i) \setminus \{u_i\}$ ,

$$\deg_{H_i}(u_i) \geq (r + s - 1)\deg_{H_i}(u) + \binom{r-1}{2} + (s+3)(r-1),$$

we have

$$\deg_{H_i}(u_i) \geq (r + s - 1) \left\lceil \frac{2L^B}{r-1} \right\rceil + \binom{r-1}{2} + (s+3)(r-1).$$

Thus, we have

$$L^B \leq \frac{1}{2} \left( \deg_{H_i}(u_i) - \binom{r-1}{2} - (s+3)(r-1) \right).$$

Note that red color can be assigned to at most one edge between any two consecutive edge of  $G$  in the ordering  $\sigma$ . Define  $l^R$  to be the number of red edges incident to  $u_i$  in the subgraph  $H_i$ . One can see that  $l^R \leq L^B + 1$ . If there exists a  $1 \leq j \leq l$  such that  $u_i \in K_j$  and  $u_r \notin K_j$  for any  $r < i$ , then set  $H'_i = H_i \cup K_j$ ; otherwise, define  $H'_i = H_i$ . Now, we show that the number of red edges incident to  $u_i$  in the graph  $H'_i$  is at most  $L^B + 1 + \frac{1}{2}\deg_{H'_i}(u_i) - \frac{1}{2}\deg_{H_i}(u_i)$ . If  $H'_i = H_i$ , then there is nothing to prove. Otherwise, one can check that red color can be assigned to at most  $\frac{1}{2}\deg_{H'_i}(u_i) - \frac{1}{2}\deg_{H_i}(u_i) + 1$  edges incident to  $u_i$  in the subgraph  $H'_i \setminus H_i$ . Moreover, if  $l^R = L^B + 1$ , then one can see that there are two red edges  $e$  and  $e'$  in  $H_i$  such that  $e$  (resp.  $e'$ ) appears before (resp. after) any edge of  $H_i^B$  in the ordering  $\sigma$ . Consequently, in this case, red color can be assigned to at most  $\frac{1}{2}\deg_{H'_i}(u_i) - \frac{1}{2}\deg_{H_i}(u_i)$  edges incident to  $u_i$  in the subgraph  $H'_i \setminus H_i$ . Thus, the number of red edges incident

to  $u_i$  in the graph  $H'_i$  is at most  $L^B + 1 + \frac{1}{2}\deg_{H'_i}(u_i) - \frac{1}{2}\deg_{H_i}(u_i)$ . Moreover, in view of the ordering of the Eulerian tour of  $\bar{H}$ , one can conclude that red color can be assigned to at most  $\frac{1}{2}(\deg_G(u_i) + s + 1 - \deg_{H'_i}(u_i))$  edges incident to  $u_i$  in the subgraph  $G \setminus H'_i$ . Therefore, the number of red edges incident to  $u_i$  in the graph  $G$  is at most

$$L^B + 1 + \frac{1}{2}\deg_{H'_i}(u_i) - \frac{1}{2}\deg_{H_i}(u_i) + \frac{1}{2}(\deg_G(u_i) + s + 1 - \deg_{H'_i}(u_i)) \leq \frac{1}{2}(\deg_G(u_i) - (s + 3)(r - 2) - \binom{r-1}{2})$$

Consequently,

$$\begin{aligned} |E(G^R)| &\leq \frac{1}{2}(\deg_G(u_i) - (s + 3)(r - 2) - \binom{r-1}{2}) + \sum_{u \in T^R \setminus \{u_i\}} \frac{1}{2}(\deg_G(u) + s + 3) \\ &\leq -\frac{1}{2}\binom{r-1}{2} + \sum_{u \in T^R} \frac{1}{2}\deg_G(u) \\ &\leq \frac{1}{2}\text{ex}(G, rK_2) \end{aligned}$$

and it is a contradiction. Hence,  $T^R = T^B$ . Therefore, the number of blue and red edges of  $G$ , i.e.,  $|E(G^R)| + |E(G^B)|$ , is at most the number of edges incident with vertices in  $T^R$  or  $T^B$  which is at most  $\text{ex}(G, rK_2)$  and this is impossible. Accordingly,  $\text{ex}_{alt}(G, rK_2, \sigma) = \text{ex}(G, rK_2)$ .  $\blacksquare$

Assume that  $G$  is a graph. A  $G$ -decomposition of a graph  $H$  is a set  $\{G_1, \dots, G_t\}$  of pairwise edge-disjoint subgraphs of  $H$  such that for each  $1 \leq i \leq t$ , the graph  $G_i$  is isomorphic to  $G$ ; and moreover, the edge sets of  $G_i$ 's partition the edge set of  $H$ . A  $G$ -decomposition of  $H$  is called a *monogamous  $G$ -decomposition*, if any distinct pair of vertices of  $H$  appear in at most one copy of  $G$  in the decomposition. Note that if a graph  $H$  has a decomposition into the complete graph  $K_t$ , then it is a monogamous  $K_t$ -decomposition.

**Theorem B.** [16] *Assume that  $m$  and  $n$  are positive even integers. The complete bipartite graph  $K_{m,n}$  has a monogamous  $C_4$ -decomposition if and only if  $(m, n) = (2, 2)$  or  $6 \leq n \leq m \leq 2n - 2$ .*

**Lemma 3.** *Let  $r, t$  and  $t'$  be positive integers, where  $11 \leq t \leq t' \leq 2t - 2$ . If  $c$  is a nonnegative integer and  $t \geq 8r + 4c + 2$ , then the complete bipartite graph  $K_{t,t'}$  is  $(r, c)$ -locally Eulerian.*

**Proof.** Assume that  $t = 2p + q$  and  $t' = 2p' + q'$ , where  $0 \leq q \leq 1$  and  $0 \leq q' \leq 1$ . Extend the complete bipartite graph  $G = K_{t,t'}$  to the complete bipartite graph  $H = K_{T,T'}$ , where  $T = t + q$  and  $T' = t' + q'$ . In view of Theorem B, consider a monogamous  $C_4$ -decomposition of  $H$ . Call any  $C_4$  of this decomposition a block, if it is entirely in  $G$ . Construct a bipartite graph with the vertex set  $(U, V)$  such that  $U$  consists of  $\lfloor \frac{t-3}{8} \rfloor$  copies of each vertex of  $K_{t,t'}$  and  $V$  consists of all blocks. Join a vertex of  $U$  to a vertex of  $V$ , if the corresponding vertex of  $K_{t,t'}$  is contained in the

corresponding block. One can check that the degree of each vertex in the part  $U$  is at least  $\lceil \frac{t-3}{2} \rceil$  and the degree of any vertex in the part  $V$  is  $4\lfloor \frac{t-3}{8} \rfloor$ . In view of Hall's Theorem, one can see that this bipartite graph has a matching which saturates all vertices of  $U$ . For any vertex  $v \in K_{t,t'}$ , consider  $\lceil \frac{t-3}{8} \rceil$  blocks assigned to  $v$  through the perfect matching and set  $H_v$  to be a subgraph of  $K_{t,t'}$  formed by the union of these blocks. One can see that  $\deg_{H_v}(v) = 2\lfloor \frac{t-3}{8} \rfloor$ , while the degree of any other vertices of  $H_v$  is 2. In view of  $H_v$ 's, one can see that  $K_{t,t'}$  is an  $(r, c)$ -locally Eulerian graph. ■

For a family of graphs  $\mathcal{F}$ , we say a graph  $G$  has an  $\mathcal{F}$ -factor if there are vertex-disjoint subgraphs  $H_1, H_2, \dots, H_t$  of  $G$  such that each  $H_i$  is a member of  $\mathcal{F}$  and  $\bigcup_{i=1}^t V(H_i) = V(G)$ . Note that if a graph  $G$  has an  $\mathcal{F}$ -factor, where each member of  $\mathcal{F}$  is an  $(r, c)$ -locally Eulerian graph, then  $G$  is also  $(r, c)$ -locally Eulerian. In view of the aforementioned lemma, if a graph  $G$  has a  $K_{t,t'}$  factor, then one can conclude that  $G$  is  $(r, c)$ -locally Eulerian. Now, we introduce some sufficient condition for a graph to have a  $K_{t,t'}$  factor.

Graph expansion was studied extensively in the literature. Let  $0 < \nu \leq \tau < 1$  and assume that  $G$  is a graph with  $n$  vertices. For  $S \subseteq V(G)$ , the  $\nu$ -robust neighborhood of  $S$ ,  $RN_{\nu, G}(S)$ , is the set of all vertices  $v \in V(G)$  such that  $|N_G(v) \cap S| \geq \nu n$ . A graph  $G$  is called *robust  $(\nu, \tau)$ -expander*, if for any  $S \subseteq V(G)$  with  $\tau n \leq |S| \leq (1-\tau)n$  we have  $|RN_{\nu, G}(S)| \geq |S| + \nu n$ . Throughout this section, we write  $0 < a \ll b \ll c$  to mean that we can choose the constants  $a$ ,  $b$ , and  $c$  from right to left. More precisely, there are two increasing functions  $f$  and  $g$  such that, given  $c$ , whenever we choose some  $b \leq g(c)$  and  $a \leq f(b)$ , for more about robust  $(\nu, \tau)$ -expander see [10]. A graph  $G$  with  $n$  vertices has *bandwidth* at most  $b$  if there exists a bijective assignment  $l : V(G) \rightarrow [n]$  such that for every edge  $uv \in E(G)$ , we have  $|l(u) - l(v)| \leq b$ .

**Theorem C.** [10] *Let  $\nu, \tau$ , and  $\eta$  be real numbers, where  $0 < \nu \leq \tau \ll \eta < 1$ , and  $\Delta$  be a positive integer. There exist constants  $\beta > 0$  and  $n_0$  such that the following holds. Suppose that  $H$  is a bipartite graph on  $n \geq n_0$  vertices with  $\Delta(H) \leq \Delta$  and bandwidth at most  $\beta n$ . If  $G$  is a robust  $(\nu, \tau)$ -expander with  $n$  vertices and  $\delta(G) \geq \eta n$ , then  $G$  contains a copy of  $H$ .*

**Theorem 1.** *Let  $\nu, \tau$  and  $\eta$  be real numbers, where  $0 < \nu \leq \tau \ll \eta < 1$ . Suppose that  $G$  is a robust  $(\nu, \tau)$ -expander graph on  $n$  vertices with  $\delta(G) \geq \eta n$ . If  $n$  is sufficiently large, then  $\chi(KG(G, rK_2)) = \chi_{alt}(KG(G, rK_2)) = |E(G)| - \text{ex}(G, rK_2)$ .*

**Proof.** In view of Lemma 2, it is sufficient to show that the graph  $G$  is an  $(r, c)$ -locally Eulerian graph, where  $c = \binom{r-1}{2} + 3(r-1)$ . Set  $t = 2r^2 + 14r - 6$  and let  $k$  and  $t'$  be integers such that  $n = 2tk + t'$  and  $2t \leq t' \leq 4t - 1$ . Now, set  $H$  to be a bipartite graph on  $n$  vertices with  $1 + \frac{n-t'}{2t}$  connected components such that one component is isomorphic to  $K_{\lceil \frac{t'}{2} \rceil, \lfloor \frac{t'}{2} \rfloor}$  and any other component is isomorphic to  $K_{t,t}$ . In view of Theorem C, if  $n$  is sufficiently large, then  $H$  is a spanning subgraph of  $G$ . By Lemma 3, one can see that  $K_{\lceil \frac{t'}{2} \rceil, \lfloor \frac{t'}{2} \rfloor}$  and  $K_{t,t}$  are both  $(r, c)$ -locally Eulerian graph; consequently, by Lemma 2, the theorem follows. ■

**Lemma B.** [13] Assume that  $\tau$  and  $\eta$  are positive constants, where  $\tau \ll \eta < 1$ . Let  $G$  be a graph with  $n$  vertices and degree sequence  $d_1 \leq d_2 \leq \dots \leq d_n$  such that for any  $i < \frac{n}{2}$  either  $d_i \geq i + \eta n$  or  $d_{n-i-\lfloor \eta n \rfloor} \geq n - i$ . If  $n$  is sufficiently large, then  $\delta(G) \geq \eta n$  and  $G$  is a robust  $(\tau^2, \tau)$ -expander.

In view of the previous lemma and Theorem 1, we have the next corollary.

**Corollary 1.** Let  $\gamma > 0$  be a real number. Assume that  $G$  is a connected graph with  $n$  vertices and degree sequence  $d_1 \leq d_2 \leq \dots \leq d_n$ . Also, suppose that for each  $i < \frac{n}{2}$ , we have either  $d_i \geq i + \gamma n$  or  $d_{n-i-\lfloor \gamma n \rfloor} \geq n - i$ . If  $n$  is sufficiently large, then  $\chi(\text{KG}(G, rK_2)) = |E(G)| - \text{ex}(G, rK_2)$ .

For a graph property  $\mathcal{P}$ , we say  $G(n, p)$  possesses  $\mathcal{P}$  *asymptotically almost sure*, or a.a.s. for brevity, if the probability that  $G \in G(n, p)$  possesses the property  $\mathcal{P}$  tends to 1 as  $n$  tends to infinity. For constants  $0 < \nu \ll \tau \ll p < 1$ , a.a.s. any graph  $G$  in  $G(n, p)$  is a robust  $(\nu, \tau)$ -expander graph with minimum degree at least  $\frac{\nu n}{2}$  and maximum degree at most  $2np$ . This observation and Theorem 1 imply that a.a.s. for any graph  $G$  in  $G(n, p)$  we have  $\chi(\text{KG}(G, rK_2)) = |E(G)| - \text{ex}(G, rK_2)$ . Moreover, Huang, Lee, and Sudakov [9] proved a more general theorem. Here, we state it in a special case.

**Theorem D.** [9] For positive integers  $r, \Delta$  and reals  $0 < p \leq 1$  and  $\gamma > 0$ , there exists a constant  $\beta > 0$  such that a.a.s., any spanning subgraph  $G'$  of any  $G \in G(n, p)$  with minimum degree  $\delta(G') \geq p(\frac{1}{2} + \gamma)n$  contains every  $n$ -vertex bipartite graph  $H$  which has the maximum degree at most  $\Delta$  and bandwidth at most  $\beta n$ .

In view of the proof of Theorem 1 and the previous theorem, we have the following result.

**Corollary 2.** If  $0 < p \leq 1$  and  $\gamma > 0$ , then a.a.s. for any spanning subgraph  $G'$  of any  $G \in G(n, p)$  with minimum degree at least  $p(\frac{1}{2} + \gamma)n$  we have  $\chi(\text{KG}(G', rK_2)) = |E(G')| - \text{ex}(G', rK_2)$ .

Assume that  $H$  is a graph with  $h$  vertices and  $\chi(H) = l$ . Set  $\text{cr}(H)$  to be the size of the smallest color class over all proper  $l$ -colorings of  $H$ . The *critical chromatic number*,  $\chi_{\text{cr}}(H)$ , is defined as  $(l - 1) \frac{h}{h - \text{cr}(H)}$ . One can check that  $\chi(H) - 1 < \chi_{\text{cr}}(H) \leq \chi(H)$  and equality holds in the upper bound if and only if in any  $l$ -coloring of  $H$ , all color classes have the same size. Assume that  $H$  has  $k$  connected components  $C_1, C_2, \dots, C_k$ . Define  $\text{hcf}_c(H)$  to be the highest common factor of integers  $|C_1|, |C_2|, \dots, |C_k|$ . Let  $f$  be an  $l$ -coloring of  $H$  such that  $x_1 \leq x_2 \leq \dots \leq x_l$  are the size of coloring classes in  $f$ . Set  $D(f) = \{x_{i+1} - x_i \mid 1 \leq i \leq l - 1\}$  and  $D(H) = \bigcup D(f)$  where the union ranges over all  $l$ -colorings  $f$  of  $H$ . Now, define  $\text{hcf}_\chi(H)$  to be the highest common factor of the members of  $D(H)$ . If  $D(H) = \{0\}$ , then we define  $\text{hcf}_\chi(H) = \infty$ . We say

$$H \text{ is in class 1 if } \begin{cases} \text{hcf}_\chi(H) = 1 & \text{when } \chi(H) \neq 2, \\ \text{hcf}_\chi(H) \leq 2 \text{ and } \text{hcf}_c(H) = 1 & \text{when } \chi(H) = 2, \end{cases}$$

otherwise,  $H$  is in Class 2.



**Theorem E.** [12] *For every graph  $H$  on  $h$  vertices, there are integers  $c$  and  $m_0$  such that for all integers  $m \geq m_0$ , if  $G$  is a graph on  $n = mh$  vertices, then the following holds. If*

$$\delta(G) \geq \begin{cases} (1 - \frac{1}{\chi_{\text{cr}}(H)})n + c & H \text{ is in Class 1,} \\ (1 - \frac{1}{\chi(H)})n + c & H \text{ is in Class 2,} \end{cases}$$

*then  $G$  has an  $H$ -factor.*

**Theorem 2.** *For any integer  $r \geq 2$ , there are constants  $\alpha$  and  $\beta$  such that for any graph  $G$  with  $n$  vertices, if  $\delta(G) \geq (\frac{1}{2} - \alpha)n + \beta$  and  $n$  is sufficiently large, then  $\chi(\text{KG}(G, rK_2)) = \chi_{\text{alt}}(\text{KG}(G, rK_2)) = |E(G)| - \text{ex}(G, rK_2)$ .*

**Proof.** Define  $t = 2r^2 + 14r - 6$ . Set  $H$  to be a bipartite graph with two connected components  $C_1$  and  $C_2$  isomorphic to  $K_{t,t}$  and  $K_{t+1,t}$ , respectively. One can check that  $H$  is in Class 1. Hence, by Theorem E, there are integers  $c_1$  and  $m_1$  such that if  $|V(G')| \geq m_1$ ,  $|V(H)|$  divides  $|V(G')|$ , and  $\delta(G') \geq (1 - \frac{1}{\chi_{\text{cr}}(H)})|V(G')| + c_1$ , then the graph  $G'$  has an  $H$ -factor. Let  $T$  be an integer such that  $4t + 1 \leq T < 8t + 2$  and  $4t + 1 | n - T$ . It is known that if  $n$  is sufficiently large and  $\delta(G) \geq \eta n$ , where  $0 < \eta < 1$ , then  $G$  contains a copy of the complete bipartite graph  $K_{\lceil \frac{T}{2} \rceil, \lfloor \frac{T}{2} \rfloor}$ . Note that  $\frac{1}{\chi_{\text{cr}}(H)} = \frac{1}{2} + \frac{1}{8t+2}$ . Set  $\alpha = \frac{1}{8t+2}$  and  $\beta = c_1 + 8t - 1$ . If  $\delta(G) \geq (\frac{1}{2} - \alpha)n + \beta$  and  $n$  is sufficiently large, then  $G$  contains  $K_{\lceil \frac{T}{2} \rceil, \lfloor \frac{T}{2} \rfloor}$  and also the graph  $G \setminus K_{\lceil \frac{T}{2} \rceil, \lfloor \frac{T}{2} \rfloor}$  has an  $H$ -factor. Hence,  $G$  can be decomposed into the complete bipartite graphs  $K_{t,t}$ ,  $K_{t+1,t}$ , and  $K_{\lceil \frac{T}{2} \rceil, \lfloor \frac{T}{2} \rfloor}$ . In view of Lemma 3, these graphs are  $(r, c)$ -locally Eulerian graphs with  $c = \binom{r-1}{2} + 3(r-1)$ . Therefore,  $G$  is an  $(r, c)$ -locally Eulerian graph; and consequently, by Lemma 2, the assertion holds. ■

### 3.2 Permutation Graphs

Assume that  $m, n, r$  are positive integers, where  $r \leq m, n$ . For an  $r$ -subset  $A \subseteq [m]$  and an injective map  $f : A \rightarrow [n]$ , the ordered pair  $(A, f)$  is said to be an  $r$ -partial permutation [5]. Let  $S_r(m, n)$  denotes the set of all  $r$ -partial permutations. Two partial permutations  $(A, f)$  and  $(B, g)$  are said to be intersecting, if there exists an  $x \in A \cap B$  such that  $f(x) = g(x)$ . Note that  $S_n(n, n)$  is the set of all  $n$ -permutations. The permutation graph  $S_r(m, n)$  has all  $r$ -partial permutations  $(A, \sigma)$  as its vertex set and two  $r$ -partial permutations are adjacent if and only if they are not intersecting. Note that  $S_r(m, n) \cong S_r(n, m)$ ; and therefore, for the simplicity, we assume that  $m \geq n$  for all permutation graphs. One can see that the permutation graph  $S_r(m, n)$  is isomorphic to  $\text{KG}(K_{m,n}, rK_2)$ .

Next theorem gives a sufficient condition for a balanced bipartite graph to have a decomposition into complete bipartite subgraphs.

**Theorem F.** [26] *For any integer  $q \geq 2$ , there exists a positive integer  $m_0$  such that for all  $m \geq m_0$ , the following holds. If  $G = (X, Y)$  is a balanced bipartite graph on  $2n = 2mq$  vertices, i.e.,  $|X| = |Y| = mq$ , with*

$$\delta(G) \geq \begin{cases} \frac{n}{2} + q - 1 & \text{if } m \text{ is even} \\ \frac{n+3q}{2} - 2 & \text{if } m \text{ is odd,} \end{cases}$$

*then  $G$  has a  $K_{q,q}$ -factor.*

Now, we investigate the chromatic number of general Kneser graph  $\text{KG}(G, rK_2)$  provided that  $G = (X, Y)$  is a balanced ( $|X| = |Y|$ ) dense bipartite graph. In particular, we determine the chromatic number of any permutation graph provided that the number of its vertices is large enough. For more about permutation graphs, see [3, 6, 11].

**Theorem 3.** *For any positive integer  $r$ , there exist constants  $q$  and  $m$  such that for all  $n \geq m$ , the following holds. If  $G$  is a graph on  $2n$  vertices which has a bipartite subgraph  $H = (U, V)$  with  $|U| = |V| = n$  and  $\delta(H) \geq \frac{n}{2} + q$ , then  $\chi(\text{KG}(G, rK_2)) = \chi_{\text{alt}}(\text{KG}(G, rK_2)) = |E(G)| - \text{ex}(G, rK_2)$ .*

**Proof.** In view of Lemma 2, it is sufficient to show that the graph  $H$  is an  $(r, c)$ -locally Eulerian graph, where  $c = \binom{r-1}{2} + 3(r-1)$ . Set  $t = 2r^2 + 14r - 6$ . By Theorem F, there are integers  $q_1$  and  $m_1$  such that if  $n \geq m_1$  and  $t|n$ , then any balanced bipartite graph  $H'$  with  $2n$  vertices and  $\delta(H') \geq \frac{n}{2} + q_1$  has a  $K_{t,t}$ -factor.

Let  $t'$  be an integer, where  $t \leq t' < 2t$  and  $t|n - t'$ . It is known that if  $n$  is sufficiently large and  $\delta(H) \geq \eta n$ , where  $0 < \eta < 1$ , then  $H$  contains a copy of the complete bipartite graph  $K_{t',t'}$ . Define  $q = q_1 + 2t - 1$ . Note that if  $n$  is sufficiently large, then  $H$  contains a copy of  $K_{t',t}$  and also, in view of Theorem F,  $H \setminus K_{t',t}$  has a  $K_{t,t}$ -factor. This implies that  $H$  can be decomposed into complete bipartite graphs  $K_{t',t'}$  and  $K_{t,t}$ . In view of Lemma 3, these graphs are  $(r, c)$ -locally Eulerian graphs, where  $c = \binom{r-1}{2} + 3(r-1)$ . Therefore,  $H$  and  $G$  are  $(r, c)$ -locally Eulerian graphs; and consequently, by Lemma 2, the assertion holds. ■

**Corollary 3.** *Assume that  $m, n, r$  are positive integers, where  $m \geq n \geq r$ . If  $m$  is large enough, then*

$$\chi(\text{KG}(K_{m,n}, rK_2)) = \chi_{\text{alt}}(\text{KG}(K_{m,n}, rK_2)) = m(n - r + 1).$$

**Proof.** In view of Hall's Theorem, any maximal  $rK_2$ -free subgraph of  $K_{m,n}$  has  $(r-1)m$  edges. Hence, in view of Lemma A, we have  $\chi(\text{KG}(K_{m,n}, rK_2)) \leq m(n - r + 1)$ . In view of Lemma 3, if  $m$  is sufficiently large, then  $\chi(\text{KG}(K_{m,m}, rK_2)) = m(m - r + 1)$ . Now, we show that for any positive integer  $n < m$ , if  $m$  is sufficiently large, then  $\chi(\text{KG}(K_{m,n}, rK_2)) = m(n - r + 1)$ . To see this, on the contrary, suppose that  $f : V(\text{KG}(K_{m,n}, rK_2)) \rightarrow \{1, 2, \dots, \chi(\text{KG}(K_{m,n}, rK_2))\}$  is a proper coloring of  $\text{KG}(K_{m,n}, rK_2)$ , where  $\chi(\text{KG}(K_{m,n}, rK_2)) < m(n - r + 1)$ . Add  $m - n$  new vertices to the small part of  $K_{m,n}$  and join them to all vertices in the other part to construct  $K_{m,m}$ , and call the new edges  $e_1, \dots, e_{(m-n)m}$ . Extend the coloring  $f$  to a proper coloring  $g$  for  $\text{KG}(K_{m,m}, rK_2)$  as follows. If a matching  $M$  is a subset of  $K_{m,n}$ , then set  $g(M) = f(M)$ ; otherwise, assume that  $i$  is the smallest positive integer such that  $e_i \in M$ , in this case set  $g(M) = i + \chi(\text{KG}(K_{m,n}, rK_2))$ . This provides a proper coloring for  $\text{KG}(K_{m,m}, rK_2)$  with less than  $m(m - r + 1)$  colors which is a contradiction. ■

The chromatic number of permutation graph  $\text{KG}(G, rK_2)$  was studied in [2] and it was proved that if  $G$  is a connected  $(s, t)$ -regular bipartite graph with  $s \geq t$  and even  $s$ , then  $\chi(\text{KG}(G, rK_2)) = |E(G)| - \text{ex}(G, rK_2)$ . This result shows that  $\chi(\text{KG}(K_{m,n}, rK_2)) = S_r(m, n) = m(n - r + 1)$  provided that  $m$  is an even integer

and  $m \geq n \geq r$ . However, if  $m$  is a small odd value, then the chromatic number of the permutation graph  $S_r(m, n)$  is unknown. The aforementioned results motivate us to consider the following conjecture.

**Conjecture 1.** *For any connected graph  $G$  and positive integer  $r$ , we have*

$$\chi(\text{KG}(G, rK_2)) = |E(G)| - \text{ex}(G, rK_2).$$

## 4 Hedetniemi's Conjecture

In this section, first we study the alternating chromatic number of the Mycielski construction of graphs. Then, we present some lower bounds for the chromatic number in terms of alternating chromatic number and strong alternating chromatic number of the Categorical product of graphs. In view of bounds, we determine the chromatic number of the Categorical product of some families of graphs.

In the sequel, we generalize the definitions of alternating and strong alternating chromatic number of graphs. We can reformulate several well-known concepts in graph colorings via graph homomorphism. One can see that if two graph are homomorphically equivalent, then they have the same chromatic number, circular chromatic number, and fractional chromatic number. In this regard, we define the following generalizations of the alternating chromatic number and the strong alternating chromatic number of a graph  $G$ , respectively, as follows

$$\chi_{\text{halt}}(G) = \max \{ \chi_{\text{alt}}(L); G \longleftrightarrow L \},$$

$$\chi_{\text{hsalt}}(G) = \max \{ \chi_{\text{salt}}(L); G \longleftrightarrow L \}.$$

Clearly, we have

$$\chi(G) \geq \max \{ \chi_{\text{halt}}(G), \chi_{\text{hsalt}}(G) \} \geq \max \{ \chi_{\text{alt}}(G), \chi_{\text{salt}}(G) \}.$$

We do not know how much  $\chi_{\text{halt}}(G)$  can be far apart  $\chi_{\text{alt}}(G)$ ; although, it seems that  $\chi_{\text{halt}}(G) = \chi_{\text{alt}}(G)$ . It is worth noting that we can sometimes easily determine  $\chi_{\text{halt}}(G)$  rather than  $\chi_{\text{alt}}(G)$ , see Lemma 4.

### 4.1 Mycielski Construction

For a given graph  $G$  with the vertex set  $V(G) = \{u_1, \dots, u_n\}$ , the *Mycielskian* of  $G$  is the graph  $M(G)$  with the vertex set  $V(M(G)) = \{u_1, \dots, u_n, v_1, \dots, v_n, w\}$  and the edge set  $E(M(G)) = E(G) \cup \{u_i v_j : u_i u_j \in E(G)\} \cup \{w v_i : 1 \leq i \leq n\}$ . In fact, the *Mycielski* graph  $M(G)$  contains the graph  $G$  itself as an isomorphic induced subgraph, together with  $n + 1$  additional vertices. For any  $1 \leq i \leq n$ ,  $v_i$  is called the twin of  $u_i$  and they have the same neighborhood in  $G$  and also the vertex  $w$  is termed the root vertex. The vertex  $w$  and  $v_i$ 's form a star graph. Some coloring properties of the Mycielski graph  $M(G)$  have been studied in the literature. For instance, it is known that  $\chi(M(G)) = \chi(G) + 1$  and  $\chi_f(M(G)) = \chi_f(G) + \frac{1}{\chi_f(G)}$ .

For any vector  $\vec{r} = (r_1, \dots, r_n)$ , the  $\vec{r}$ -blow up graph  $G(\vec{r})$  of  $G$  is obtained by replacing each vertex  $u_i$  of  $G$  by  $r_i$  copies  $u_i^1, \dots, u_i^{r_i}$ , such that for any  $1 \leq i_1 \leq r_i$

and  $1 \leq j_1 \leq r_j$ ,  $u_i^{j_1}$  is adjacent to  $u_j^{j_1}$  if  $u_i$  is adjacent to  $u_j$  in  $G$ . In other words, the edge  $u_i u_j$  is replaced by the complete bipartite graph  $K_{r_i, r_j}$ . Note that for any vector  $\vec{r}$  with positive entries, two graphs  $G(\vec{r})$  and  $G$  are homomorphically equivalent.

**Lemma 4.** *For any graph  $G$ , we have  $\chi_{\text{halt}}(M(G)) \geq \chi_{\text{halt}}(G) + 1$ .*

**Proof.** Let  $\mathcal{F} \subseteq 2^{[n]}$  and that  $\text{KG}(\mathcal{F})$  is homomorphically equivalent to  $G$  such that there exists an ordering  $\sigma$  of  $[n]$  for which  $\chi_{\text{halt}}(G) = n - \text{alt}_\sigma(\mathcal{F})$ . Assume that  $\mathcal{F} = \{A_1, \dots, A_m\}$ . Set  $\vec{r} = (r_1, \dots, r_{2m+1})$ , where  $r_1 = \dots = r_m = 2t + 1$ ,  $r_{m+1} = \dots = r_{2m} = \binom{2t+1}{t+1}$ ,  $r_{2m+1} = 1$  and  $t = n - \text{alt}_\sigma(\mathcal{F})$ . In what follows, we introduce a Kneser representation for the  $\vec{r}$ -blow up of  $M(\text{KG}(\mathcal{F}))$ , i.e.,  $M(\text{KG}(\mathcal{F}))(\vec{r})$ . Note that  $M(\text{KG}(\mathcal{F}))(\vec{r})$  and  $M(G)$  are homomorphically equivalent. Define

$$V' = \{b_1, b_2, \dots, b_{2t+1}, c_1, c_2, \dots, c_{(2t+1)(m-1)}\},$$

$$V'' = \{a_{1,1}, a_{1,2}, \dots, a_{1,(2t+1)}, a_{2,1}, a_{2,2}, \dots, a_{2,(2t+1)}, \dots, a_{m,1}, a_{m,2}, \dots, a_{m,2t+1}\},$$

where  $V'$ ,  $V''$ , and the set  $[n]$  are pairwise disjoint. Set  $V = [n] \cup V' \cup V''$  and  $l = \binom{2t+1}{t+1}$ . For any  $1 \leq i \leq m$  and  $1 \leq j \leq 2t + 1$ , define  $A_{i,j} = A_i \cup \{a_{i,j}\}$ . Moreover, for any  $1 \leq i \leq m$ , consider distinct sets  $B_{i,1}, \dots, B_{i,l}$  such that for any  $1 \leq k \leq l$ , there exists a unique  $(t+1)$ -subset  $\{b_{k_1}, b_{k_2}, \dots, b_{k_{t+1}}\}$  of  $\{b_1, b_2, \dots, b_{2t+1}\}$  where  $B_{i,k} = A_i \cup \{b_{k_1}, b_{k_2}, \dots, b_{k_{t+1}}\}$ . Set

$$H = \{A_{i,j}, B_{i,k} : 1 \leq i \leq m, 1 \leq j \leq 2t + 1, 1 \leq k \leq l\} \cup \{V''\}.$$

Now, one can check that  $\text{KG}(H)$  provides a Kneser representation for  $M(\text{KG}(\mathcal{F}))(\vec{r})$ . To see this, one can check that  $A_{ij}$ 's,  $B_{ij}$ 's, and  $V''$  are corresponding to the vertices of  $\text{KG}(\mathcal{F})$ , their twins, and the root vertex, respectively. Note that  $H \subseteq 2^V$  and  $c_1, \dots, c_{(m-1)(2t+1)}$  are the isolated vertices of  $H$ . As a benefit of using isolated vertices, we present an ordering  $\pi$  to determine the alternating chromatic number of  $M(\text{KG}(\mathcal{F}))(\vec{r})$ . First, consider the ordering  $\tau$  as follows

$$\begin{aligned} a_{1,1} &< c_1 < a_{2,1} < c_2 < \dots < a_{m-1,1} < c_{m-1} < a_{m,1} < b_1 < \\ a_{1,2} &< c_m < a_{2,2} < c_{m+1} < \dots < a_{m-1,2} < c_{2m-2} < a_{m,2} < b_2 < \\ &\vdots \\ a_{1,2t+1} &< c_{2t(m-1)+1} < a_{2,2t+1} < \dots < c_{(2t+1)(m-1)} < a_{m,2t+1} < b_{2t+1} \end{aligned}$$

Construct the ordering  $\pi$  by concatenating the ordering  $\sigma$  after  $\tau$ , i.e.,  $\pi = \tau \parallel \sigma$ . Note that the number of elements of  $\pi$  is  $(2t+1)m + (2t+1) + (2t+1)(m-1) + n = 2m(2t+1) + n$ . Define  $p = 2m(2t+1) + n$ . Now, we claim that  $\text{alt}_\pi(H) \leq \text{alt}_\sigma(\mathcal{F}) + 2m(2t+1) - 1$ . To see this, assume that  $X = (x_1, x_2, \dots, x_p) \in \{-1, 0, +1\}^p \setminus \{(0, 0, \dots, 0)\}$  and  $\text{alt}(X) = \text{alt}_\sigma(\mathcal{F}) + 2m(2t+1)$ . We show that  $X_\pi^+$  or  $X_\pi^-$  contains a hyperedge of  $H$ . If  $\text{alt}((x_1, x_2, \dots, x_{2m(2t+1)})) = 2m(2t+1)$ , then  $X_\pi^+$  or  $X_\pi^-$  contains  $\{a_{1,1}, a_{1,2}, \dots, a_{m,2t+1}\}$ , i.e., the root vertex; and consequently, the assertion follows. Hence, let  $\text{alt}((x_1, x_2, \dots, x_{2m(2t+1)})) \leq 2m(2t+1) - 1$ ; and consequently,  $\text{alt}(Y) \geq \text{alt}_\sigma(\mathcal{F}) + 1$ , where  $Y = (x_{2m(2t+1)+1}, x_{2m(2t+1)+2}, \dots, x_p)$ . Hence,  $Y_\sigma^+$  or  $Y_\sigma^-$  contains a hyperedge of  $\mathcal{F}$ . Without loss of generality, suppose that  $Y_\sigma^+$  contains  $A_i \in \mathcal{F}$ . If  $a_{i,j} = +1$  for some  $1 \leq j \leq 2t+1$ , then  $A_{i,j} \in H$  and  $A_{i,j} \subseteq X_\pi^+$ .

Moreover, if there exists a  $(t+1)$ -subset  $\{b_{k_1}, b_{k_2}, \dots, b_{k_{t+1}}\}$  such that  $b_{k_j} = +1$  for any  $1 \leq j \leq t+1$ , then  $A_i \cup \{b_{k_1}, b_{k_2}, \dots, b_{k_{t+1}}\} \in H$  and  $A_i \cup \{b_{k_1}, b_{k_2}, \dots, b_{k_{t+1}}\} \subseteq X_\pi^+$ . On the other hand, if for any  $1 \leq j \leq 2t+1$ ,  $a_{i,j} \neq +1$ , and if for any  $(t+1)$ -subset  $\{b_{k_1}, b_{k_2}, \dots, b_{k_{t+1}}\}$ , there exists some  $1 \leq j \leq t+1$  such that  $b_{k_j} \neq +1$ , then one can conclude that  $\text{alt}((x_1, x_2, \dots, x_{2m(2t+1)})) \leq 2m(2t+1) - (t+1) = 2m(2t+1) - n + \text{alt}_\sigma(\mathcal{F}) - 1$ . On the other hand,  $\text{alt}(X) = \text{alt}_\sigma(\mathcal{F}) + 2m(2t+1)$ ; hence, we should have  $\text{alt}(Y) \geq n+1$  which is impossible.  $\blacksquare$

A graph  $G$  is said to be *alternatively  $t$ -chromatic* (resp. *strongly alternatively  $t$ -chromatic*) if  $\chi_{\text{halt}}(G) = \chi(G) = t$  (resp.  $\chi_{\text{hsalt}}(G) = \chi(G) = t$ ). Note that if graphs  $G$  and  $H$  are homomorphically equivalent, then  $M(G)$  and  $M(H)$  are homomorphically equivalent as well. In other words, the previous theorem states that the graph  $M(G)$  is alternatively  $(t+1)$ -chromatic graph provided that  $G$  is alternatively  $t$ -chromatic graph.

## 4.2 Chromatic Number of Categorical Product

There are several kinds of graph products in the literature. The *Categorical Product*  $G \times H$  of two graphs  $G$  and  $H$  is defined by  $V(G \times H) = V(G) \times V(H)$ , where two vertices  $(u_1, u_2)$  and  $(v_1, v_2)$  are adjacent if  $u_1 v_1 \in E(G)$  and  $u_2 v_2 \in E(H)$ . It is easy to check that by any coloring of  $G$  or  $H$  we can present a coloring of  $G \times H$ ; and therefore,  $\chi(G \times H) \leq \min\{\chi(G), \chi(H)\}$ . In 1966, Hedetniemi [7] introduced his interesting conjecture, called *Hedetniemi's conjecture*, about the chromatic number of the Categorical product of two graphs, which states that  $\chi(G \times H) = \min\{\chi(G), \chi(H)\}$ . This conjecture has been studied in the literature, see [14, 19, 23, 25, 29]. In view of topological bounds for chromatic number, it was shown that Hedetniemi's conjecture holds for any two graphs for which the topological bound on the chromatic number is tight, see [8, 22, 25].

**Theorem G.** [8] *If  $G$  and  $H$  are two graphs for which the topological bound on the chromatic number is tight, then  $\chi(G \times H) = \min\{\chi(G), \chi(H)\}$ .*

Also, in [27], Hedetniemi's conjecture was generalized to circular chromatic number of graphs. Zhu [27] conjectured that  $\chi_c(G \times H) = \min\{\chi_c(G), \chi_c(H)\}$ . For more results about this conjecture and circular chromatic number, one can refer to [15, 24, 27, 28]. In this section, we present some lower bounds for the chromatic number of the Categorical product of graphs.

**Lemma 5.** *Let  $G$  and  $H$  be two graphs. If there exists a graph homomorphism  $f : H \rightarrow G$ , then  $\chi_{\text{halt}}(H) \leq \chi_{\text{halt}}(G)$  and also,  $\chi_{\text{hsalt}}(H) \leq \chi_{\text{hsalt}}(G)$ .*

**Proof.** We prove  $\chi_{\text{halt}}(H) \leq \chi_{\text{halt}}(G)$  and similarly one can show  $\chi_{\text{shalt}}(H) \leq \chi_{\text{shalt}}(G)$ . First, we assume that  $H$  is a subgraph of  $G$  and we prove a stronger assertion. In fact, we show that if  $H$  is a subgraph of  $G$ , then  $\chi_{\text{alt}}(H) \leq \chi_{\text{alt}}(G)$ . To see this, assume that  $\mathcal{F} \subseteq 2^{[n]}$  such that  $\text{KG}(\mathcal{F})$  is isomorphic to  $H$ . We show that there are  $Y$  and  $\mathcal{G} \subseteq 2^Y$  such that  $\text{KG}(\mathcal{G})$  is isomorphic to  $G$  and that  $n - \text{alt}(\mathcal{F}) \leq |Y| - \text{alt}(\mathcal{G})$ .

Without loss of generality, suppose that  $\text{alt}(\mathcal{F}) = \text{alt}_I(\mathcal{F})$ , where  $I$  is the identity ordering, i.e.,  $I : 1 < 2 < \dots < n$ . Let  $g : V(H) \rightarrow \mathcal{F}$  be an isomorphism between  $\text{KG}(\mathcal{F})$  and  $H$ . Also, we can assume that  $H$  is a spanning subgraph of  $G$ . To see this, let  $v \in V(G) \setminus V(H)$ . Now, add this vertex to  $H$  as an isolated vertex to obtain  $H_1$ . Set  $\mathcal{F}_1 = \mathcal{F} \cup \{\{1, 2, \dots, n+1\}\} \subseteq 2^{[n+1]}$ . One can see that  $\text{KG}(\mathcal{F}_1)$  is isomorphic to  $H_1$ . Also,  $\text{alt}_I(\mathcal{F}_1) \leq \text{alt}_I(\mathcal{F}) + 1$ ; and therefore,  $n+1 - \text{alt}(\mathcal{F}_1) \geq n+1 - \text{alt}_I(\mathcal{F}_1) \geq n - \text{alt}_I(\mathcal{F}) = n - \text{alt}(\mathcal{F})$ . By repeating the previous procedure, if it is necessary, we can find a spanning subgraph  $\bar{H}$  of  $G$  and a Kneser representation  $\text{KG}(\mathcal{H})$  for  $\bar{H}$  with  $\mathcal{H} \subseteq 2^{[n+l]}$  and  $l = |V(G)| - |V(H)|$  such that  $n - \text{alt}(\mathcal{F}) \leq n+l - \text{alt}(\mathcal{H})$ . Thus, it is enough to prove the lemma just for spanning subgraphs.

Now, we can assume that  $H$  is a spanning subgraph of  $G$ . Again, without loss of generality, we can assume that there is an edge  $e = ab \in E(G)$  such that  $H + e = G$ . Assume that  $g(a) = A$  and  $g(b) = B$ . Since  $a$  and  $b$  are not adjacent,  $A \cap B$  is not an empty set. Assume that  $A \cap B = \{y_1, y_2, \dots, y_t\}$ . Consider  $2t$  positive integers  $\{y'_1, y'_2, \dots, y'_t\} \cup \{z_1, z_2, \dots, z_t\}$  disjoint from  $[n]$ . Let  $\mathcal{F}_0 = \mathcal{F}$ ,  $\sigma_0 = I$ ,  $g_0 = g$ , and  $Y_i = [n] \cup \{y'_1, z_1, y'_2, z_2, \dots, y'_i, z_i\}$ . Assume that  $\mathcal{F}_i \subseteq 2^{Y_i}$ ,  $\sigma_i \in S_{Y_i}$ , and  $g_i : V(H) \rightarrow \mathcal{F}_i$  were defined when  $i < t$ . Set  $g_{i+1}(a) = g_i(a) \cup \{y'_{i+1}\} \setminus \{y_{i+1}\}$  and for  $u \notin \{a, b\}$ , if  $y_{i+1} \in g_i(u)$ , then  $g_{i+1}(u) = g_i(u) \cup \{y'_{i+1}\}$ . Let  $\mathcal{F}_{i+1} = \{g_{i+1}(v) : v \in V(H)\}$ . To obtain the ordering  $\sigma_{i+1}$ , replace  $y_{i+1}$  with  $y_{i+1} < z_{i+1} < y'_{i+1}$  in the ordering  $\sigma_i$ , i.e., put  $z_{i+1}$  immediately after  $y_{i+1}$  and put  $y'_{i+1}$  after  $z_{i+1}$ . Note that  $\mathcal{F}_{i+1} \subseteq 2^{Y_{i+1}}$  and  $\sigma_{i+1} \in S_{Y_{i+1}}$ , where  $Y_{i+1} = [n] \cup \{y'_1, z_1, y'_2, z_2, \dots, y'_{i+1}, z_{i+1}\}$ . Assume that  $\mathcal{F}_i$ 's and  $\sigma_i$ 's were obtained for every  $i \in \{0, 1, \dots, t\}$ . One can see that  $\text{alt}_{\sigma_{i+1}}(\mathcal{F}_{i+1}) \leq 2 + \text{alt}_{\sigma_i}(\mathcal{F}_i)$ ; and therefore,  $\text{alt}_{\sigma_t}(\mathcal{F}_t) \leq 2t + \text{alt}_{\sigma_0}(\mathcal{F}_0)$ . Note that  $\text{KG}(\mathcal{F}_t) \cong H + e = G$ . Set  $\mathcal{G} = \mathcal{F}_t$  and  $Y_t = Y$ . Consequently,

$$|Y| - \text{alt}(\mathcal{G}) \geq n + 2t - \text{alt}_{\sigma_t}(\mathcal{G}) \geq n + 2t - (2t + \text{alt}_{\sigma_0}(\mathcal{F}_0)) = n - \text{alt}_I(\mathcal{F}) = n - \text{alt}(\mathcal{F})$$

Hence, if  $H$  is a subgraph of  $G$ , then we have  $\chi_{\text{alt}}(H) \leq \chi_{\text{alt}}(G)$ . Similarly, one can show that  $\chi_{\text{slat}}(H) \leq \chi_{\text{salt}}(G)$ .

Now, assume that for two given graphs  $G$  and  $H$ , there exists a graph homomorphism  $f : H \rightarrow G$ . Consider  $\bar{H}$  and  $\mathcal{F} \subseteq 2^{[n]}$  such that  $\bar{H}$  is homomorphically equivalent to  $H$ ,  $\text{KG}(\mathcal{F}) \cong \bar{H}$  and  $\chi_{\text{halt}}(H) = n - \text{alt}(\mathcal{F})$ . Since  $H$  and  $\bar{H}$  are homomorphically equivalent and there exists a graph homomorphism  $f : H \rightarrow G$ , the graph  $G \cup \bar{H}$ , i.e., the disjoint union of  $G$  and  $\bar{H}$ , is homomorphically equivalent to the graph  $G$  and also, this graph has  $\bar{H}$  as its subgraph. In view of the aforementioned discussion, there are  $Y$  and  $\mathcal{G} \subseteq 2^Y$  such that  $\text{KG}(\mathcal{G}) \cong G \cup \bar{H}$  and

$$\chi_{\text{halt}}(H) = n - \text{alt}(\mathcal{F}) \leq |Y| - \text{alt}(\mathcal{G}) \leq \chi_{\text{halt}}(G \cup \bar{H}) = \chi_{\text{halt}}(G).$$

Similarly, one can show  $\chi_{\text{shalt}}(H) \leq \chi_{\text{shalt}}(G)$ . ■

In view of the proof of the aforementioned lemma, the next corollary follows.

**Corollary 4.** *If  $H$  is a subgraph of  $G$ , then  $\chi_{\text{alt}}(H) \leq \chi_{\text{alt}}(G)$  and  $\chi_{\text{slat}}(H) \leq \chi_{\text{salt}}(G)$ . In particular, for any graph  $G$ , we have  $\min\{\chi_{\text{alt}}(G), \chi_{\text{salt}}(G)\} \geq \omega(G)$ , where  $\omega(G)$  is the clique number of  $G$ .*



In the next result, we show the accuracy of Hedetniemi's conjecture for the strong alternating chromatic number of graphs.

**Lemma 6.** *For any two graphs  $G$  and  $H$ , we have*

- a)  $\chi_{hsalt}(G \times H) = \min\{\chi_{hsalt}(G), \chi_{hsalt}(H)\}$ ,
- b)  $\chi_{halt}(G \times H) \geq \max\{\min\{\chi_{halt}(G), \chi_{hsalt}(H) - 1\}, \min\{\chi_{hsalt}(G) - 1, \chi_{halt}(H)\}\}$ .

**Proof.** First, we prove part (a). Assume that  $\mathcal{G} \subseteq 2^V$  and  $\mathcal{H} \subseteq 2^{V'}$  such that  $\text{KG}(\mathcal{G})$  and  $\text{KG}(\mathcal{H})$  are homomorphically equivalent to  $G$  and  $H$ , respectively. Also, assume that  $\chi_{hsalt}(G) = 1 + |V| - \text{salt}(\mathcal{G})$ ,  $\chi_{hsalt}(H) = 1 + |V'| - \text{salt}(\mathcal{H})$ , where  $V = \{1, 2, \dots, n\}$  and  $V' = \{n+1, n+2, \dots, n+m\}$ . Without loss of generality, we can assume that  $\text{salt}(\mathcal{H}) = \text{salt}_I(\mathcal{H})$  and  $\text{salt}(\mathcal{G}) = \text{salt}_I(\mathcal{G})$ , where  $I$  is the identity ordering. Let  $\mathcal{F} = \{A \cup B : A \in \mathcal{G} \text{ \& } B \in \mathcal{H}\}$  and  $V'' = V \cup V'$ . One can check that  $\text{KG}(\mathcal{F}) \cong G \times H$  and also,  $\chi_{hsalt}(G \times H) \geq 1 + |V''| - \text{salt}(\mathcal{F}) \geq \min\{\chi_{hsalt}(G), \chi_{hsalt}(H)\}$ . To see this, it is enough to show that  $\text{salt}_I(\mathcal{F}) \leq \max\{|V'| + \text{salt}(\mathcal{G}), |V| + \text{salt}(\mathcal{H})\}$ . Define  $l = \max\{|V'| + \text{salt}(\mathcal{G}), |V| + \text{salt}(\mathcal{H})\}$ . Consider  $X \in \{-1, 0, 1\}^{n+m}$  with  $\text{alt}_I(X) \geq 1 + l$ .

Now, consider two vectors  $X(1), X(2) \in \{-1, 0, 1\}^{n+m}$  such that the first  $n$  coordinates of  $X(1)$  (resp. the last  $m$  coordinates of  $X(2)$ ) are the same as  $X$  and the last  $m$  coordinates of  $X(1)$  (resp. the first  $n$  coordinates of  $X(2)$ ) are zero. If we show that  $\text{alt}_I(X(1)) > \text{salt}(\mathcal{G})$  and  $\text{alt}_I(X(2)) > \text{salt}(\mathcal{H})$ , then each of  $X^+$  and  $X^-$  has a hyperedge of  $\mathcal{F}$  and it completes the proof. Suppose that  $\text{alt}_I(X(1)) \leq \text{salt}(\mathcal{G})$  (resp.  $\text{alt}_I(X(2)) \leq \text{salt}(\mathcal{H})$ ). Therefore,  $\text{alt}_I(X) \leq \text{alt}_I(X(1)) + \text{alt}_I(X(2)) \leq \text{salt}(\mathcal{G}) + |V'| \leq l$  (resp.  $\text{alt}_I(X) \leq \text{alt}_I(X(1)) + \text{alt}_I(X(2)) \leq |V| + \text{salt}(\mathcal{H}) \leq l$ ) which is a contradiction. Hence,  $\chi_{hsalt}(G \times H) \geq \min\{\chi_{hsalt}(G), \chi_{hsalt}(H)\}$ . On the other hand, there exists a graph homomorphism from  $G \times H$  to both  $G$  and  $H$ . Consequently, by Lemma 5, we have  $\chi_{hsalt}(G \times H) \leq \min\{\chi_{hsalt}(G), \chi_{hsalt}(H)\}$ .

Now, we prove part (b). By symmetry, it suffices to prove  $\chi_{halt}(G \times H) \geq \min\{\chi_{halt}(G), \chi_{hsalt}(H) - 1\}$ . Consider  $\mathcal{G} \subseteq 2^V$ ,  $\mathcal{H} \subseteq 2^{V'}$ ,  $\sigma \in S_V$ , and  $\gamma \in S_{V'}$  such that  $G \longleftrightarrow \text{KG}(\mathcal{G})$ ,  $H \longleftrightarrow \text{KG}(\mathcal{H})$ ,  $\chi_{halt}(G) = |V| - \text{alt}_\sigma(\mathcal{G})$ , and  $\chi_{hsalt}(H) = |V'| + 1 - \text{alt}_\gamma(\mathcal{H})$ . Without loss of generality, we can assume that  $V = \{1, 2, \dots, n\}$ ,  $V' = \{n+1, n+2, \dots, n+m\}$ ,  $\sigma : 1 < 2 < \dots < n$ , and  $\gamma : n+1 < n+2 < \dots < n+m$ . Define  $\mathcal{L} = \{A \cup B : A \in \mathcal{G} \text{ \& } B \in \mathcal{H}\} \subseteq 2^{[n+m]}$ . One can check that  $\text{KG}(\mathcal{L}) \cong \text{KG}(\mathcal{G}) \times \text{KG}(\mathcal{H})$ ; and therefore,  $\text{KG}(\mathcal{L}) \longleftrightarrow G \times H$ . Set  $I : 1 < 2 < \dots < m+n$  and  $M = \max\{|V| + \text{salt}_\gamma(\mathcal{H}), |V'| + \text{alt}_\sigma(\mathcal{G})\}$ . Now, we show that  $\text{alt}_I(\mathcal{L}) \leq M$ . To see this, assume that we have an  $X \in \{-1, 0, +1\}^{m+n} \setminus \{(0, 0, \dots, 0)\}$  such that  $\text{alt}(X) \geq M + 1$ . Let  $X(1), X(2) \in \{-1, 0, 1\}^{m+n}$  be the same as in the proof of previous part. One can see that there exists an alternative subsequence of nonzero terms in  $X(1)$  of length more than  $\text{alt}_\sigma(\mathcal{G})$ . Therefore, either  $X(1)_I^+$  or  $X(1)_I^-$  has some hyperedge of  $\mathcal{G}$ . Now, we show that both  $X(2)_I^+$  and  $X(2)_I^-$  have some hyperedges of  $\mathcal{H}$ . On the contrary, suppose that this is not true. Therefore, we have  $\text{alt}(X(2)) \leq \text{salt}_\gamma(\mathcal{H})$  and thus  $\text{alt}(X) \leq |V| + \text{alt}(X(2)) \leq M$  which is a contradiction.

Without loss of generality, we can assume that  $X(2)_I^+$  and  $X(2)_I^-$  contain  $A$  and  $B$  of  $\mathcal{H}$ , respectively, and  $X(1)_I^+$  contains  $C$  of  $\mathcal{G}$ . Now, in view of  $A \cup C \subseteq X_I^+$  and that  $A \cup C \in \mathcal{L}$ , the assertion follows.

Hence, we have

$$\begin{aligned}
\chi_{halt}(G \times H) &\geq m + n - alt_I(\mathcal{L}) \\
&\geq m + n - M \\
&= \min\{n - alt_\sigma(\mathcal{G}), m - salt_\gamma(\mathcal{H})\} \\
&= \min\{\chi_{halt}(G), \chi_{hsalt}(H) - 1\}
\end{aligned}$$

as desired. ■

**Lemma 7.** Assume that there are  $\mathcal{G} \subseteq 2^V$ ,  $\mathcal{H} \subseteq 2^{V'}$ ,  $\sigma \in S_V$ , and  $\gamma \in S_{V'}$  such that  $\chi_{alt}(\text{KG}(\mathcal{G})) = |V| - alt_\sigma(\mathcal{G})$  and  $\chi_{alt}(\text{KG}(\mathcal{H})) = |V'| - alt_\gamma(\mathcal{H})$ . If

$$\max\{|V| + alt_\gamma(\mathcal{H}), |V'| + alt_\sigma(\mathcal{G})\} \geq salt_\sigma(\mathcal{G}) + salt_\gamma(\mathcal{H}),$$

then  $\chi_{alt}(\text{KG}(\mathcal{G}) \times \text{KG}(\mathcal{H})) \geq \min\{\chi_{alt}(\text{KG}(\mathcal{G})), \chi_{alt}(\text{KG}(\mathcal{H}))\}$ .

**Proof.** Let  $G = \text{KG}(\mathcal{G})$  and  $H = \text{KG}(\mathcal{H})$ . Without loss of generality, we can assume that  $V = \{1, 2, \dots, n\}$ ,  $V' = \{n+1, n+2, \dots, n+m\}$ ,  $\sigma : 1 < 2 < \dots < n$ , and  $\gamma : n+1 < n+2 < \dots < n+m$ . Define  $\mathcal{L} = \{A \cup B : A \in \mathcal{G} \text{ \& } B \in \mathcal{H}\}$ . Note that  $\text{KG}(\mathcal{L}) \cong \text{KG}(\mathcal{G}) \times \text{KG}(\mathcal{H}) \cong G \times H$ . Set  $I : 1 < 2 < \dots < m+n$  and  $M = \max\{|V| + alt_\gamma(\mathcal{H}), |V'| + alt_\sigma(\mathcal{G})\}$ . In view of the assumption, we have

$$M = \max\{|V| + alt_\gamma(\mathcal{H}), |V'| + alt_\sigma(\mathcal{G})\} \geq salt_\gamma(\mathcal{H}) + salt_\sigma(\mathcal{G}).$$

Now, we show that  $alt_I(\mathcal{L}) \leq M$ . To see this, assume that we have an  $X \in \{-1, 0, +1\}^{m+n} \setminus \{(0, 0, \dots, 0)\}$  such that  $alt(X) \geq M + 1$ . Consider two vectors  $X(1), X(2) \in \{-1, 0, 1\}^{m+n}$  such that the first  $n$  coordinates of  $X(1)$  (resp. the last  $m$  coordinates of  $X(2)$ ) are the same as  $X$  and the last  $m$  coordinates of  $X(1)$  (resp. the first  $n$  coordinates of  $X(2)$ ) are zero. One can see that there exists an alternative subsequence of nonzero terms in  $X(1)$  (resp.  $X(2)$ ) of length more than  $alt_\sigma(\mathcal{G})$  (resp.  $alt_\gamma(\mathcal{H})$ ). Therefore, either  $X(1)_I^+$  or  $X(1)_I^-$  (resp. either  $X(2)_I^+$  or  $X(2)_I^-$ ) has some hyperedge of  $\mathcal{G}$  (resp.  $\mathcal{H}$ ). Now, we show that either both  $X(1)_I^+$  and  $X(1)_I^-$  or both  $X(2)_I^+$  and  $X(2)_I^-$  have some hyperedges of  $\mathcal{G}$  or  $\mathcal{H}$ , respectively. On the contrary, suppose that this is not true. Therefore, we have  $alt(X(1)) \leq salt_\sigma(\mathcal{G})$  and  $alt(X(2)) \leq salt_\gamma(\mathcal{H})$ . Note that these inequalities imply that  $alt(X) \leq alt(X(1)) + alt(X(2)) \leq M$  which is a contradiction.

Without loss of generality, we can suppose that  $X(1)_I^+$  and  $X(1)_I^-$  contain  $A$  and  $B$  of  $\mathcal{G}$ , respectively, and also  $X(2)_I^+$  contains  $C$  of  $\mathcal{H}$ . Now, in view of  $A \cup C \subseteq X_I^+$  and that  $A \cup C \in \mathcal{L}$ , the assertion follows.

Hence, we have

$$\begin{aligned}
\chi_{alt}(G \times H) &\geq m + n - alt_I(\mathcal{L}) \\
&\geq m + n - M \\
&= \min\{\chi_{alt}(G), \chi_{alt}(H)\}
\end{aligned}$$

as desired. ■

**Remark 2.** In view of the aforementioned lemma, one can determine the chromatic number of the Categorical product of some family of graphs. For instance,

one can consider matching-dense graphs with  $n$  vertices, where  $n$  is sufficiently large. In view of Lemma 7, it is sufficient to introduce a good upper bound for  $\text{ex}_{\text{salt}}(G, rK_2, \sigma)$ , where  $\sigma$  is the same ordering presented in the proof of Lemma 2. We show  $\text{ex}_{\text{salt}}(G, rK_2, \sigma) \leq \text{ex}(G, rK_2) + \binom{r-1}{2} + (s+3)(r-1) + 1$ . To see this, consider an alternating 2-coloring of the edges of  $G$  with respect to the ordering  $\sigma$  of length  $\text{ex}(G, rK_2) + \binom{r-1}{2} + (s+3)(r-1) + 2$  and suppose that  $G^R$  has no matching of size  $r$ . By a similar argument as in the proof of Lemma 2, one can conclude that  $|T^R| = r - 1$ . Also,

$$|E(G^R)| \leq \sum_{u \in T^R} \frac{1}{2}(\deg_G(u) + s + 3) \leq \frac{1}{2} \left( \text{ex}(G, rK_2) + \binom{r-1}{2} + (s+3)(r-1) \right)$$

which is impossible.

In view of Lemmas 6 and 7, one can determine the chromatic number of the Categorical product of some family of graphs. In particular, if both of them are strongly alternatively  $t$ -chromatic graphs. Note that the following graphs are strongly alternatively  $t$ -chromatic graphs.

1. Schrijver graphs and Kneser graphs
2. The Kneser multigraph  $\text{KG}(G, \mathcal{F})$ :  $G$  is a multigraph such that all of its edges have even multiplicities and  $\mathcal{F}$  is a family of its simple subgraphs, see [2].
3. The matching graph  $\text{KG}(G, rK_2)$ :  $G$  is a connected graph with  $n$  vertices, odd girth at least  $g$ , and degree sequence  $\deg_G(v_1) \geq \deg_G(v_2) \geq \dots \geq \deg_G(v_n)$ . Also,  $r \leq \max\{\frac{g}{2}, \frac{\deg_G(v_{r-1})+1}{4}\}$ ,  $\{v_1, \dots, v_{r-1}\}$  forms an independent set, and  $\deg_G(v_1), \deg_G(v_2), \dots, \deg_G(v_{r-1})$  are even integers, see [2].

Note that, in [22], it has been shown that for any two graphs  $G$  and  $H$  we have  $\text{coind}(B(G \times H)) = \min\{\text{coind}(G), \text{coind}(H)\}$ . Also, in [4], it was proved that for any positive integer  $r$ , we have  $\text{coind}(B(M_r(G))) \geq \text{coind}(B(G)) + 1$ , where  $M_r(G)$  is the generalized Mycielskian of  $G$ . Consequently, in view of Lemma 1, one can see that Hedetniemi's conjecture holds for any two graphs of the family of strongly alternatively  $t$ -chromatic graphs and the iterated generalized Mycielskian of any such graphs.

Also, the following graphs are alternatively  $t$ -chromatic graphs.

1. Kneser graphs and multiple Kneser graphs: In [1], multiple Kneser graphs were introduced as a generalization of Kneser graphs.
2. Kneser Multigraphs, see [2].
3. Matching-dense graphs and a family of matching-sparse graphs, see [2].
4. The permutation graph  $S_r(m, n)$ :  $m$  is large enough.
5. The Schrijver graph  $\text{SG}(n, k)$ :  $k = 2$  or  $n = 2k + 1$ .

6. Any number of iterations of the Mycielski construction starting with any graph appearing on the list above.

**Acknowledgement:** The authors would like to express their deepest gratitude to Professor Carsten Thomassen for his insightful comments. They also appreciate the detailed valuable comments of Dr. Saeed Shaebani. Moreover, they would like to thank Skype for sponsoring their endless conversations in two countries.

## References

- [1] M. Alishahi and H. Hajiabolhassan. On Chromatic Number of Kneser Hypergraphs. *ArXiv e-prints*, February 2013. [1](#), [2](#), [4](#), [6](#), [9](#), [23](#)
- [2] M. Alishahi and H. Hajiabolhassan. Chromatic Number Via Turán Number. *ArXiv e-prints*, December 2013. [1](#), [2](#), [4](#), [6](#), [7](#), [8](#), [9](#), [16](#), [23](#)
- [3] P. J. Cameron and C. Y. Ku. Intersecting families of permutations. *European J. Combin.*, 24(7):881–890, 2003. [16](#)
- [4] P. Csorba. Fold and Mycielskian on homomorphism complexes. *Contrib. Discrete Math.*, 3(2):1–8, 2008. [23](#)
- [5] X. B. Geng, J. Wang, and H. J. Zhang. Structure of independent sets in direct products of some vertex-transitive graphs. *Acta Math. Sin. (Engl. Ser.)*, 28(4):697–706, 2012. [15](#)
- [6] C. Godsil and K. Meagher. A new proof of the Erdős-Ko-Rado theorem for intersecting families of permutations. *European J. Combin.*, 30(2):404–414, 2009. [16](#)
- [7] S. Hedetniemi. *Homomorphisms of graphs and automata*. Technical report, University of Michigan, 03105–44-T, 1996. [19](#)
- [8] P. Hell. An introduction to the category of graphs. In *Topics in graph theory* (New York, 1977), volume 328 of *Ann. New York Acad. Sci.*, pages 120–136. New York Acad. Sci., New York, 1979. [2](#), [19](#)
- [9] H. Huang, C. Lee, and B. Sudakov. Bandwidth theorem for random graphs. *J. Combin. Theory Ser. B*, 102(1):14–37, 2012. [14](#)
- [10] F. Knox and A. Treglown. Embedding spanning bipartite graphs of small bandwidth. *Combin. Probab. Comput.*, 22(1):71–96, 2013. [13](#)
- [11] C. Y. Ku and I. Leader. An Erdős-Ko-Rado theorem for partial permutations. *Discrete Math.*, 306(1):74–86, 2006. [16](#)
- [12] D. Kühn and D. Osthus. The minimum degree threshold for perfect graph packings. *Combinatorica*, 29(1):65–107, 2009. [15](#)

- [13] D. Kühn, D. Osthus, and A. Treglown. Hamiltonian degree sequences in digraphs. *J. Combin. Theory Ser. B*, 100(4):367–380, 2010. [14](#)
- [14] B. Larose and C. Tardif. Hedetniemi’s conjecture and the retracts of a product of graphs. *Combinatorica*, 20(4):531–544, 2000. [19](#)
- [15] B. Larose and C. Tardif. Projectivity and independent sets in powers of graphs. *J. Graph Theory*, 40(3):162–171, 2002. [19](#)
- [16] C. C. Lindner and A. Rosa. Monogamous decompositions of complete bipartite graphs, symmetric H-squares, and self-orthogonal 1-factorizations. *Australas. J. Combin.*, 20:251–256, 1999. [12](#)
- [17] L. Lovász. Kneser’s conjecture, chromatic number, and homotopy. *J. Combin. Theory Ser. A*, 25(3):319–324, 1978. [3](#)
- [18] J. Matoušek. *Using the Borsuk-Ulam theorem*. Universitext. Springer-Verlag, Berlin, 2003. Lectures on topological methods in combinatorics and geometry, Written in cooperation with Anders Björner and Günter M. Ziegler. [5](#)
- [19] N. W. Sauer and X. Zhu. An approach to Hedetniemi’s conjecture. *J. Graph Theory*, 16(5):423–436, 1992. [19](#)
- [20] A. Schrijver. Vertex-critical subgraphs of Kneser graphs. *Nieuw Arch. Wisk.* (3), 26(3):454–461, 1978. [3](#)
- [21] G. Simonyi and G. Tardos. Local chromatic number, Ky Fan’s theorem and circular colorings. *Combinatorica*, 26(5):587–626, 2006. [5](#)
- [22] G. Simonyi and A. Zsbán. On topological relaxations of chromatic conjectures. *European J. Combin.*, 31(8):2110–2119, 2010. [2](#), [19](#), [23](#)
- [23] C. Tardif. The fractional chromatic number of the categorical product of graphs. *Combinatorica*, 25(5):625–632, 2005. [19](#)
- [24] C. Tardif. Multiplicative graphs and semi-lattice endomorphisms in the category of graphs. *J. Combin. Theory Ser. B*, 95(2):338–345, 2005. [19](#)
- [25] C. Tardif. Hedetniemi’s conjecture, 40 years later. *Graph Theory Notes N. Y.*, 54:46–57, 2008. [2](#), [19](#)
- [26] Y. Zhao. Bipartite graph tiling. *SIAM J. Discrete Math.*, 23(2):888–900, 2009. [15](#)
- [27] X. Zhu. Circular chromatic number: a survey. *Discrete Math.*, 229(1-3):371–410, 2001. [19](#)
- [28] X. Zhu. Recent developments in circular colouring of graphs. In *Topics in discrete mathematics*, volume 26 of *Algorithms Combin.*, pages 497–550. Springer, Berlin, 2006. [19](#)
- [29] X. Zhu. The fractional version of Hedetniemi’s conjecture is true. *European J. Combin.*, 32(7):1168–1175, 2011. [19](#)